Lecture 6
MATH-42021/52021 Graph Theory and Combinatorics.

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Department of Mathematical Sciences, Kent State University

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Euler Cycles (Example, instead of introduction).

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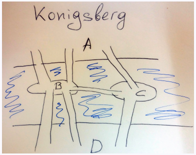

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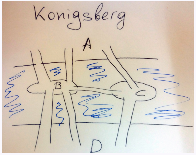

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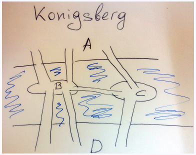

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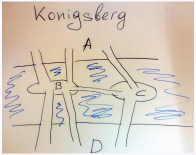

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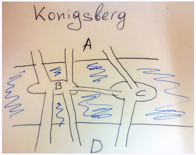

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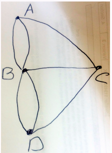

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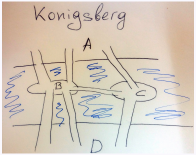

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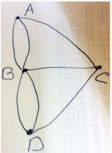

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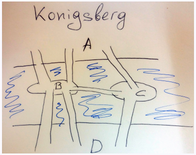

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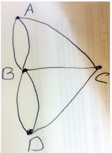

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- A **cycle** is a sequence of consecutively linked edges $((x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n))$ whose starting vertex is the ending vertex, i.e. $x_1 = x_n$ and which no edge can appear more then once.

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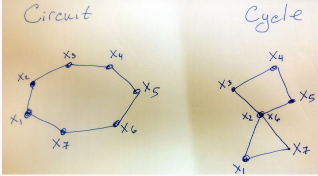

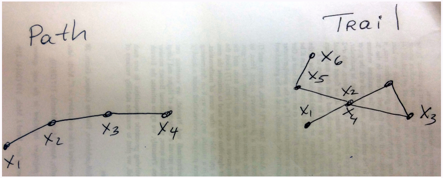

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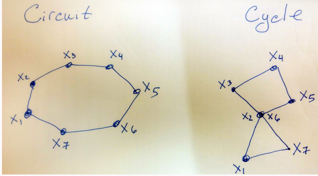

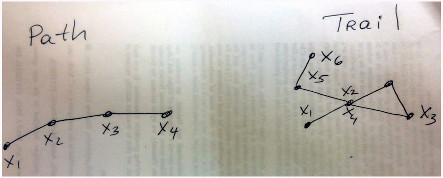

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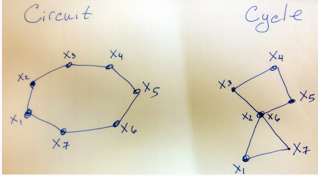

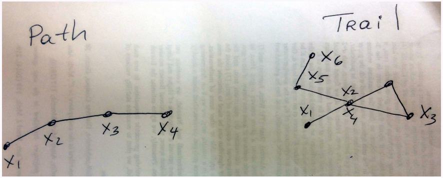

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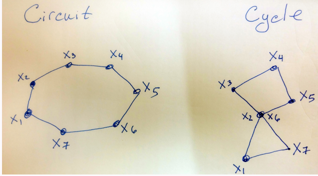

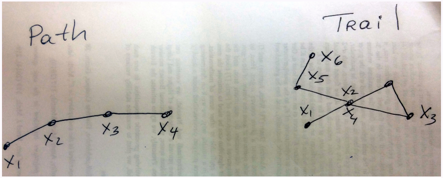

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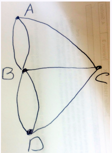
- An **Euler trail** is a trail that contains ALL the edges in a graph (and visits each vertex at least once).
- For some applications of Euler cycles we will need to allow a multiple edges between vertices as well a loops (and edge of the form (x, x)) – we will call such generalization of a graph — **"multigraphs"**.

So why there is no Euler cycle in Konigsberg?



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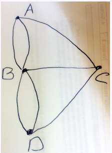

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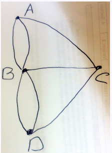

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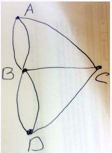

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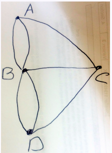

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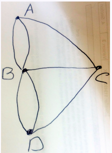

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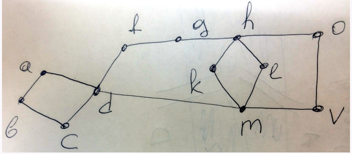

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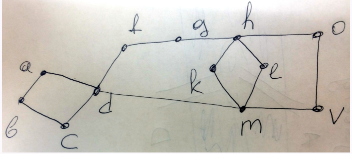

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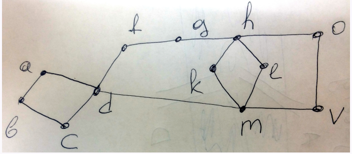

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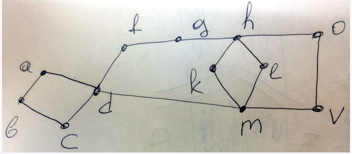

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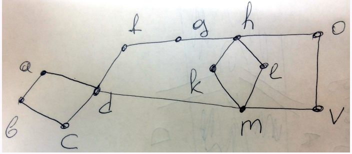

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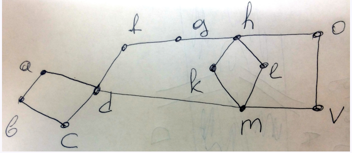

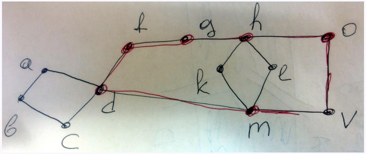

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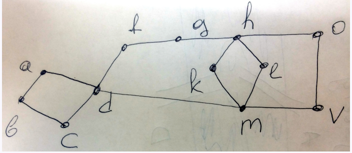

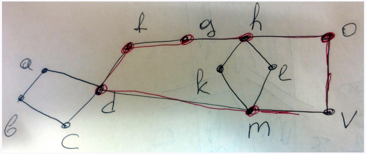

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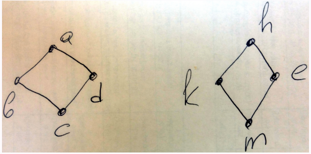

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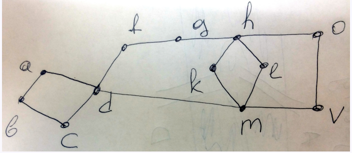

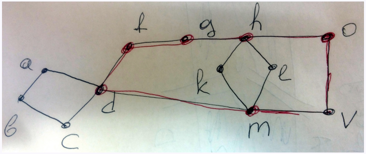

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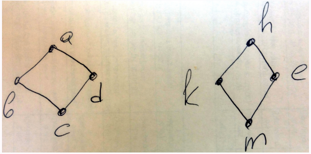

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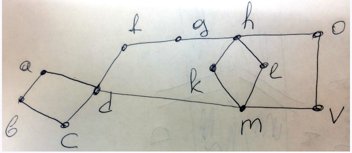

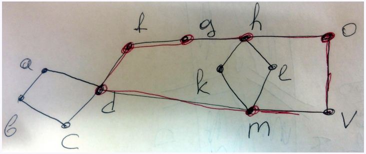

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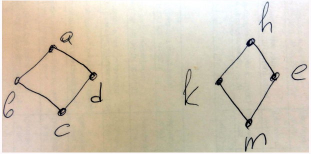

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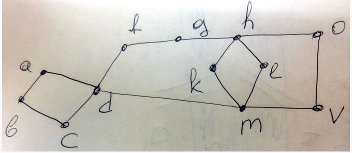

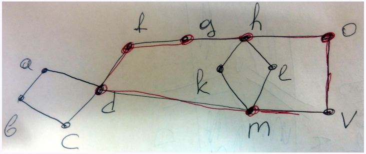

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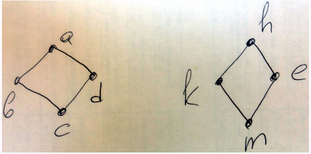

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Euler Cycles.

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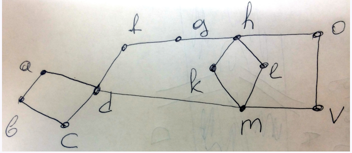

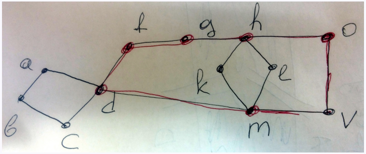

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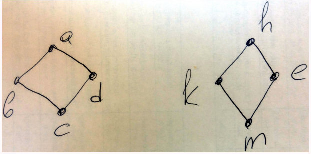

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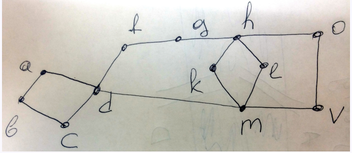

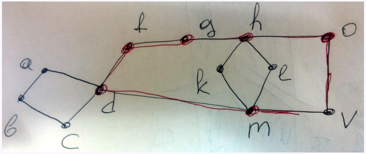

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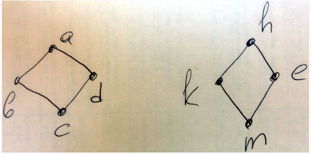

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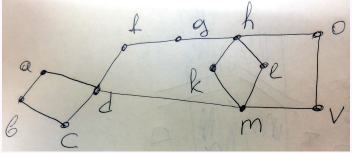

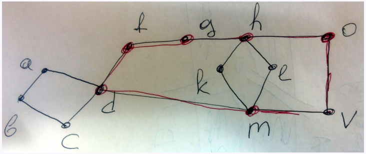

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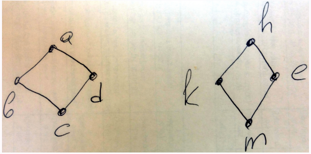

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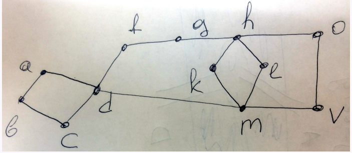

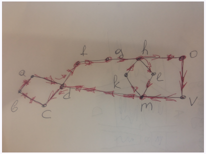

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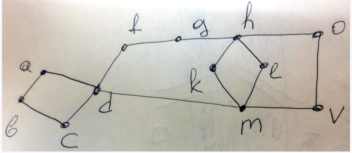

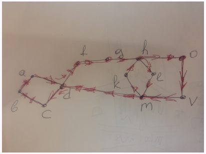

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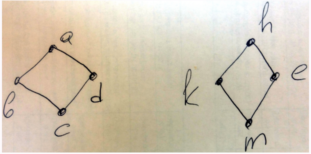

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