Lecture 8
MATH-42021/52021 Graph Theory and Combinatorics.

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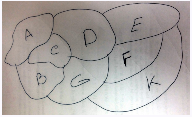
June, 2016.

Graph Coloring - A reminder

One of the oldest problems in graph theory is connected with map coloring. The question is what is the minimal number of different colors are needed to color countries on some map so that any pair of countries with a common border are given different colors.
Here an example:



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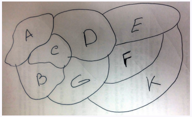

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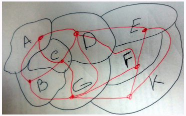

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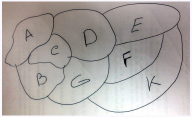

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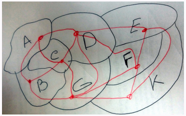

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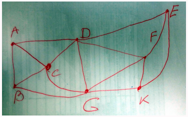

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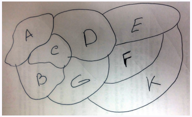
Here an example (which is planar):



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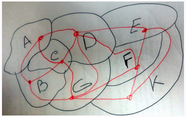

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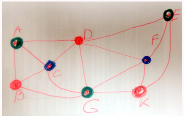

The question now is how many colors we need to "color" the vertices such that adjacent vertices have different colors. For the above graph we can do with 3 colors.

Would 3 colors be enough to color a planar graph?

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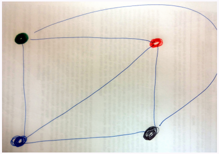

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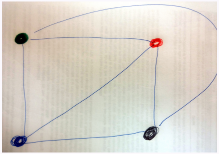

So we need at least 4 colors to color a planar graph (and thus a map).

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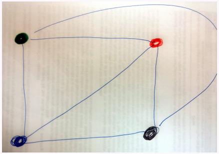

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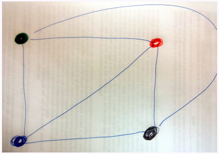

So we need at least 4 colors to color a planar graph (and thus a map). But would 4 colors be enough? The answer is YES, but this is a VERY non-trivial question which took a long time to be solved. But helped to develop a very interesting theory of planar graphs.

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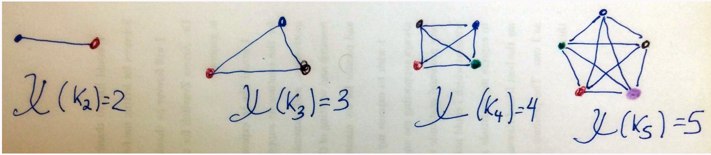

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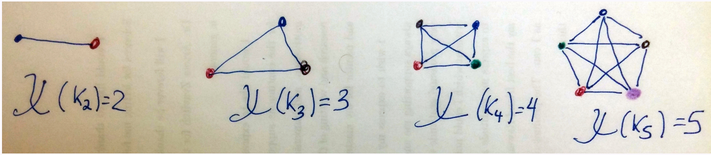

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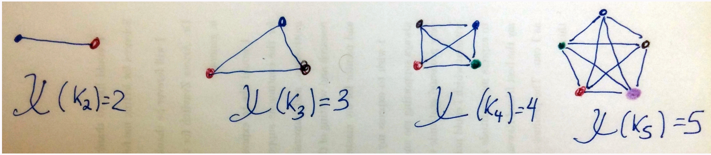

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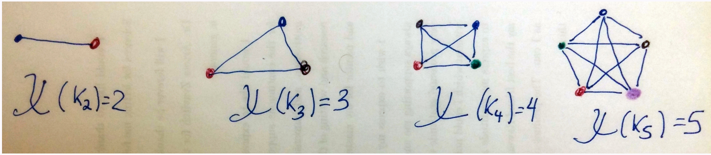

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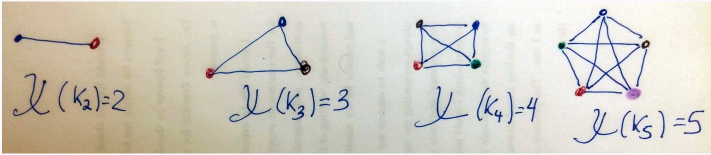

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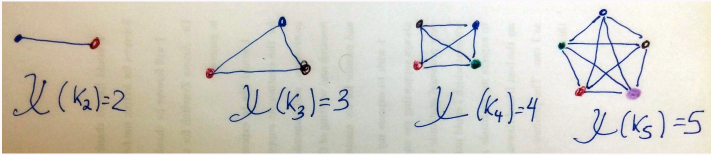

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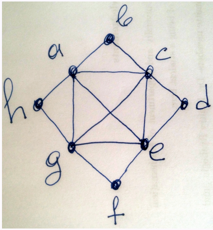

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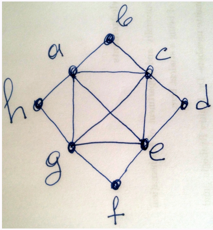

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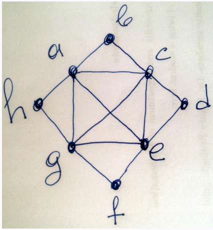

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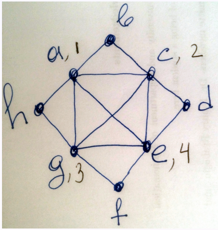

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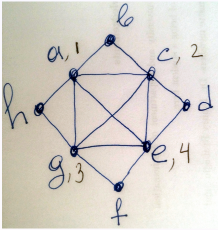

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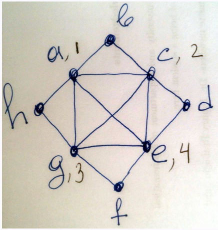

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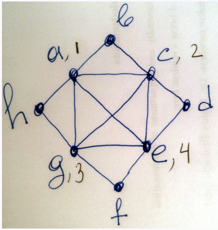

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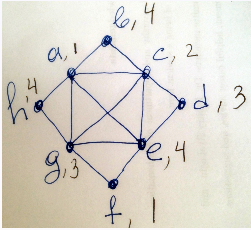

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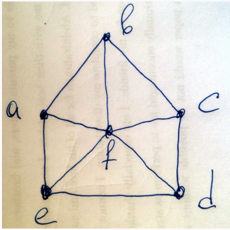

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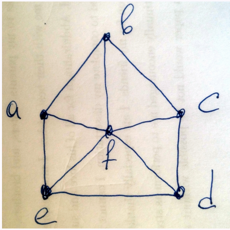

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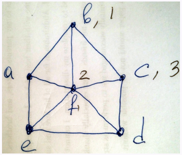

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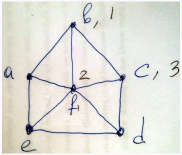

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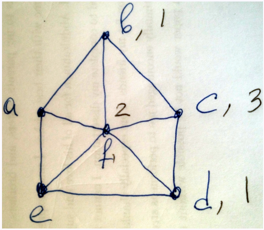

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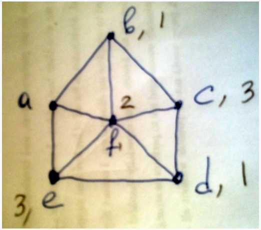

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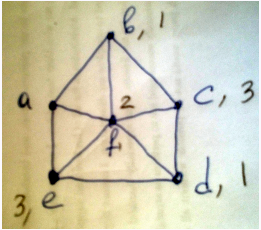

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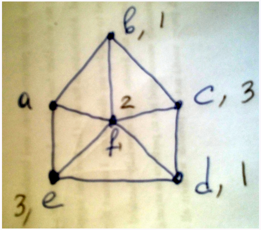

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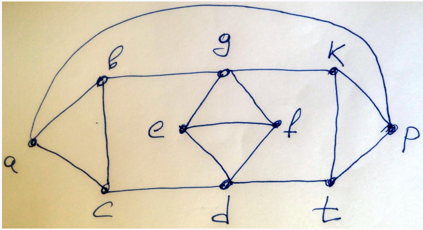

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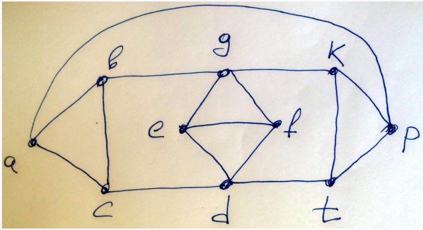

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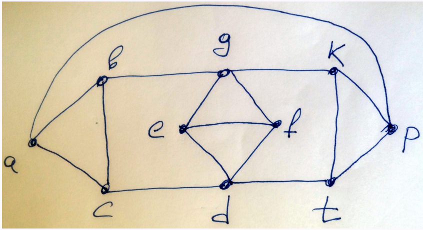

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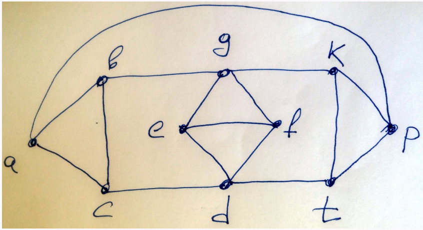

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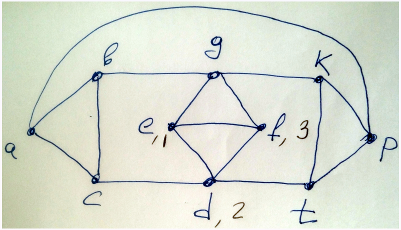

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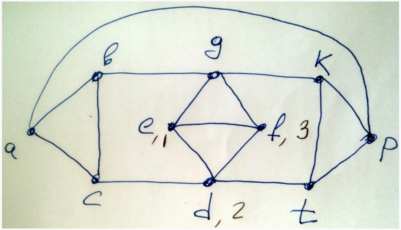

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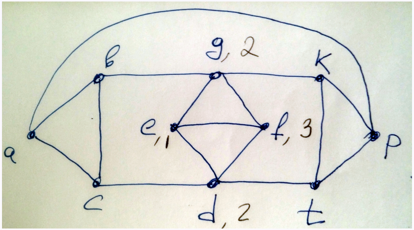

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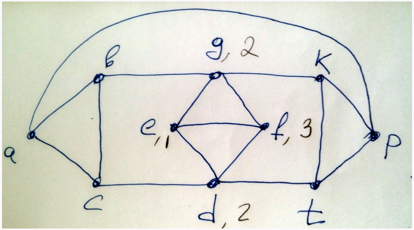

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Graph Coloring (Third Example)

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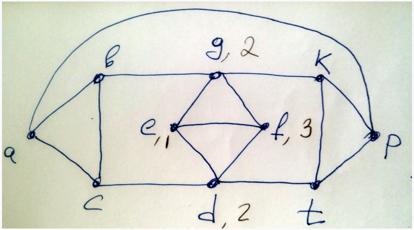

The largest complete graph that we see is again K_3, so let's start with 3 colors. The idea is to select a triangle which would force coloring to some other vertices. This is possible if we start, for example, with triangle $e, f, d \to (1, 3, 2)$. Next we notice that g must be colored in 2. It seems not clear how to continue now. But we must try! So, what is clear is that k and t are both adjacent to each other and two vertices of color 2.

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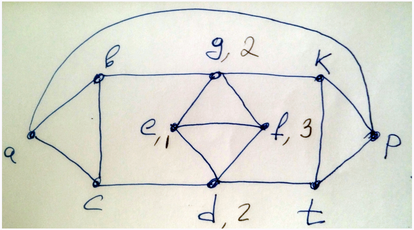

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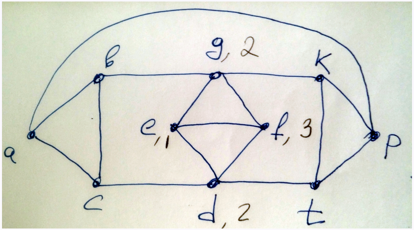

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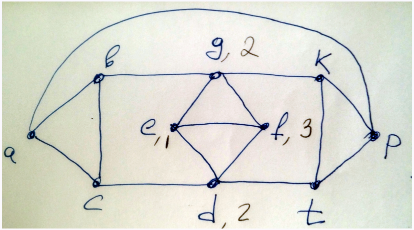

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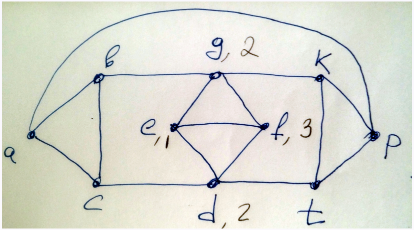

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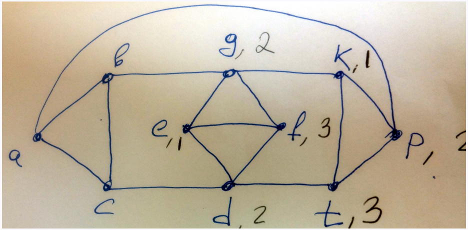

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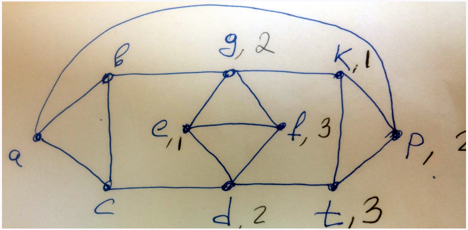

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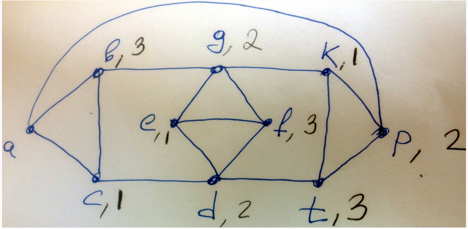

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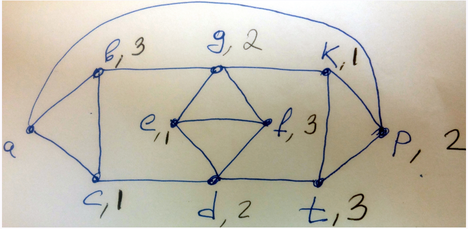

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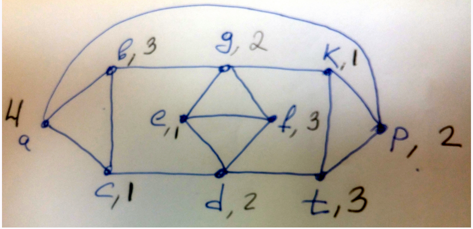

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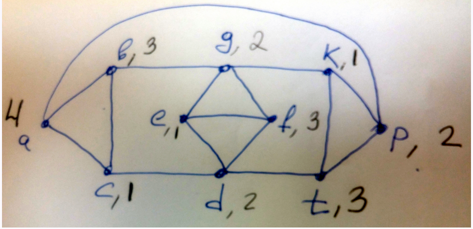

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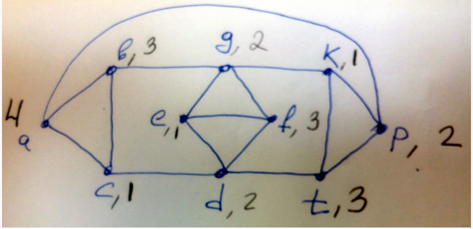

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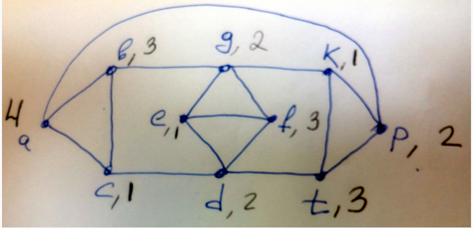

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