Lecture 9
MATH-42021/52021 Graph Theory and Combinatorics.

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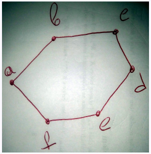

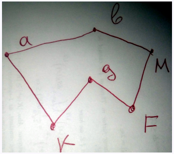

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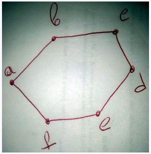

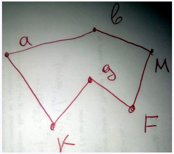

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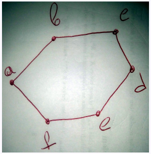

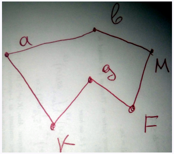

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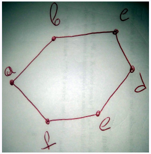

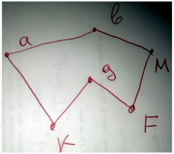

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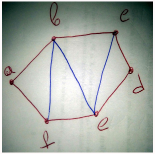

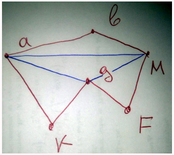

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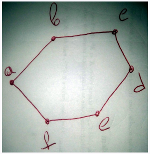

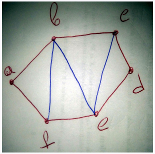

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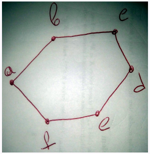

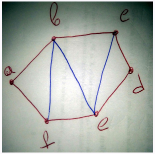

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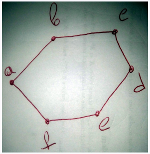

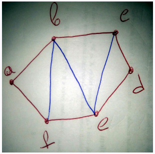

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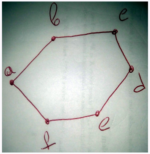

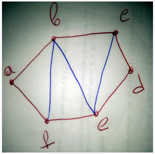

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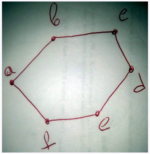

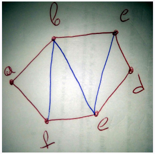

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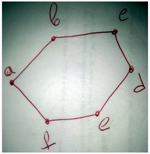

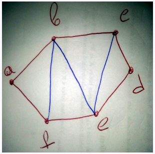

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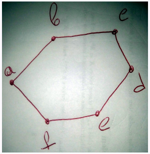

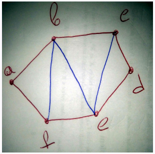

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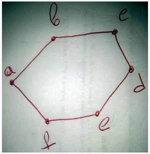

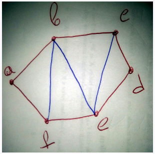

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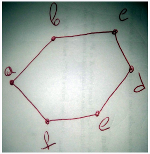

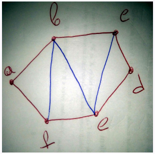

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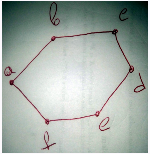
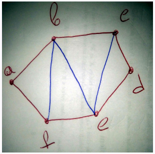

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Make a triangulation of the polygon formed by the walls of the gallery. Note that a guard at any corner of a triangle has all sides of the triangle under surveillance. Now obtain 3-coloring of triangulation (say "red, blue, green"). Each triangle will have exactly one corner which is red, one conner which is green and one conner which is blue. Now if we place a guard in each corner (vertex!) which is (for example) red we get that all wall of the gallery are watched! Now our polygon has n walls $=$ edges, thus the number of vertices is also n.

Graph Coloring - Theorem 1 - cool application

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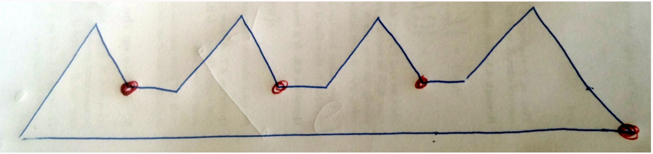
Note (using the example below) that the bound is the best possible



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We will prove a bit weaker, but still very cool theorem

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Remark: show that the above estimate is the best possible, i.e. create a planar graph such that all vertices have degree at least 5.

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So our graph is connected, there is a vertex of degree 5 or less and the statement is true if number of vertices in v is less or equal to 5.

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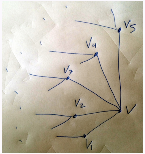

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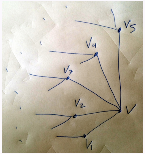

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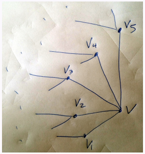

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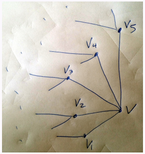

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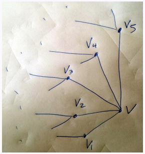

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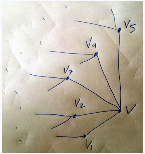

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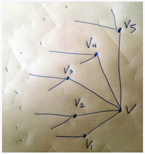

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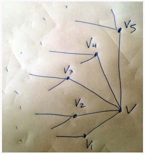

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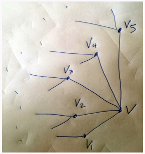

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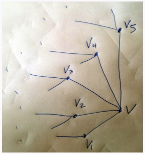

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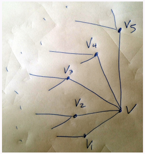

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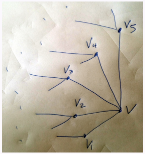

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Case 5 - such path exists: Ready for a cool trick? If such path exists there is NO way to create a path of colors 2 and 4 only, from vertex v_2 to vertex v_4 and we can change the color of 2 to 4 exactly as we done in the Case 1.

Graph Coloring -More theorems (without proofs, sorry)

Brook's Theorem

If the graph G is not an odd circuit or complete graph, then $\chi(G) \leq d$, where d is the maximum degree of a vertex in G.

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Vizing's Theorem

If the maximum degree of a vertex in a graph G is d, then the edge chromatic number of G is either d or $d+1$.