Lecture 9
MATH-42021/52021 Graph Theory and Combinatorics.

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Graph Coloring - Theorems.

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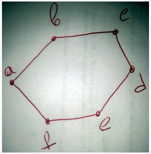

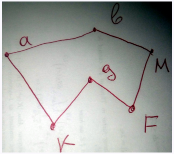

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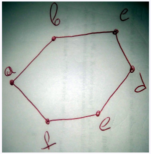

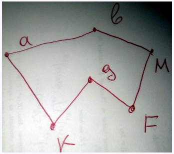

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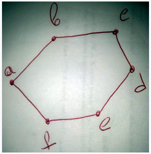

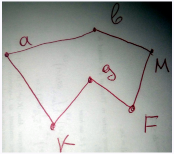

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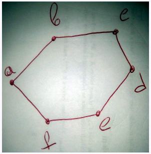

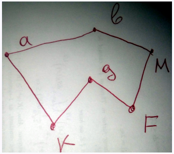

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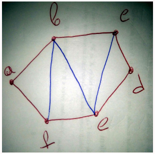

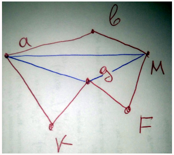

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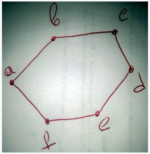

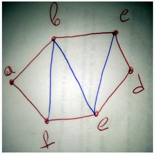

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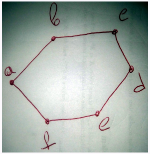

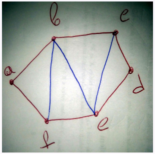

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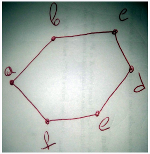

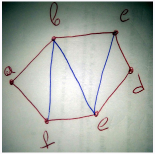

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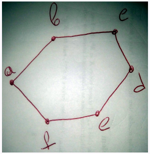

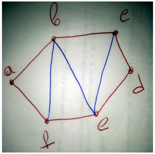

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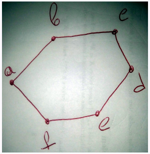 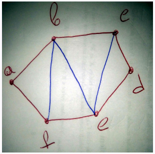

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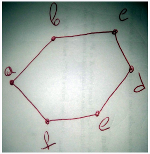

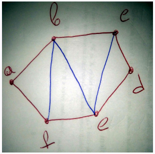

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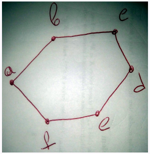

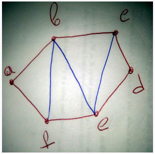

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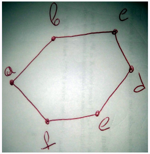

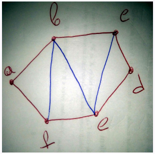

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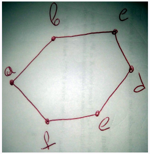

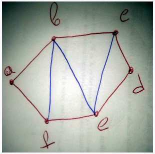

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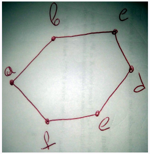

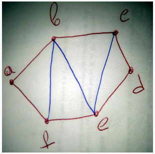

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□

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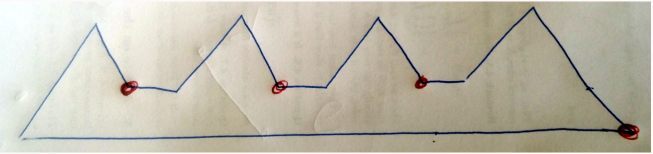
Note (using the example below) that the bound is the best possible



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We will prove a bit weaker, but still very cool theorem

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Remark: show that the above estimate is the best possible, i.e. create a planar graph such that all vertices have degree at least 5.

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So our graph is connected, there is a vertex of degree 5 or less and the statement is true if number of vertices in v is less or equal to 5.

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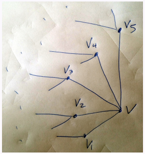

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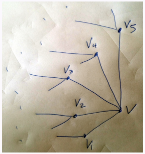

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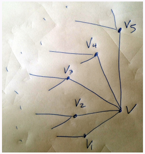

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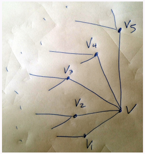

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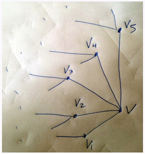

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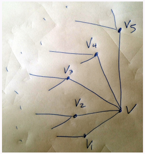

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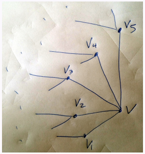

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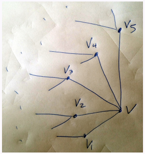

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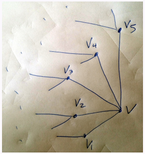

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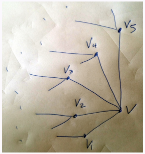

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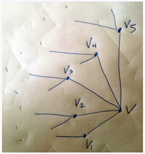

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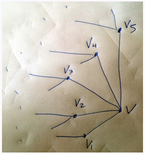

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Case 5 - such path exists: Ready for a cool trick? If such path exists there is NO way to create a path of colors 2 and 4 only, from vertex v_2 to vertex v_4 and we can change the color of 2 to 4 exactly as we done in the Case 1.

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