

On the Brunn-Minkowski inequality for general measures with applications to new isoperimetric-type inequalities

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Abstract

In this paper we present new versions of the classical Brunn-Minkowski inequality for different classes of measures and sets. We show that the inequality

$$\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda\mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}$$

holds true for an unconditional product measure μ with decreasing density and a pair of unconditional convex bodies $A, B \subset \mathbb{R}^n$. We also show that the above inequality is true for any unconditional log-concave measure μ and unconditional convex bodies $A, B \subset \mathbb{R}^n$. Finally, we prove that the inequality is true for a symmetric log-concave measure μ and a pair of symmetric convex sets $A, B \subset \mathbb{R}^2$, which, in particular, settles two-dimensional case of the conjecture for Gaussian measure proposed in [13].

In addition, we deduce the $1/n$ -concavity of the parallel volume $t \mapsto \mu(A + tB)$, Brunn's type theorem and certain analogues of Minkowski first inequality.

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1 Introduction

The classical Brunn-Minkowski inequality states that for any two non-empty compact sets $A, B \subset \mathbb{R}^n$ and any $\lambda \in [0, 1]$ we have

$$\text{vol}_n(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \text{vol}_n(A)^{1/n} + (1 - \lambda) \text{vol}_n(B)^{1/n}, \quad (1)$$

with equality if and only if $B = aA + b$, where $a > 0$ and $b \in \mathbb{R}^n$. Here vol_n stands for the Lebesgue measure on \mathbb{R}^n and

$$A + B = \{a + b : a \in A, b \in B\}$$

is the Minkowski sum of A and B . Due to homogeneity of the volume, this inequality is equivalent to $\text{vol}_n(A + B)^{1/n} \geq \text{vol}_n(A)^{1/n} + \text{vol}_n(B)^{1/n}$. The Brunn-Minkowski inequality turns out to be a powerful tool. In particular, it implies the classical isoperimetric inequality: for any compact set $A \subset \mathbb{R}^n$ we have

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$\text{vol}_n(A_t) \geq \text{vol}_n(B_t)$, $t \geq 0$, where B is a Euclidean ball satisfying $\text{vol}_n(A) = \text{vol}_n(B)$ and A_t stands for the t -enlargement of A , i.e., $A_t = A + tB_2^n$, where B_2^n is the unit Euclidean ball, $B_2^n = \{x : |x| = 1\}$. To see this it is enough to observe that

$$\text{vol}_n(A + tB_2^n)^{1/n} \geq \text{vol}_n(A)^{1/n} + \text{vol}_n(tB_2^n)^{1/n} = \text{vol}_n(B)^{1/n} + \text{vol}_n(tB_2^n)^{1/n} = \text{vol}_n(B + tB_2^n)^{1/n}.$$

Taking $t \rightarrow 0^+$ one gets a more familiar form of isoperimetry: among all sets with fixed volume the surface area

$$\text{vol}_n^+(\partial A) = \liminf_{t \rightarrow 0^+} \frac{\text{vol}_n(A + tB_2^n) - \text{vol}_n(A)}{t}$$

is minimized in the case of the Euclidean ball. We refer to [11] for more information on Brunn-Minkowski-type inequalities.

Using the inequality between means one gets an a priori weaker dimension free form of (1), namely

$$\text{vol}_n(\lambda A + (1 - \lambda)B) \geq \text{vol}_n(A)^\lambda \text{vol}_n(B)^{1-\lambda}. \quad (2)$$

In fact (2) and (1) are equivalent. To see this one has to take $\tilde{A} = A/\text{vol}_n(A)^{1/n}$, $\tilde{B} = B/\text{vol}_n(B)^{1/n}$ and $\tilde{\lambda} = \lambda \text{vol}_n(A)^{1/n}/(\lambda \text{vol}_n(A)^{1/n} + (1 - \lambda) \text{vol}_n(B)^{1/n})$ in (2). This phenomenon is a consequence of homogeneity of the Lebesgue measure.

The above notions can be generalized to the case of the so-called s -concave measures. Here we assume that $s > 0$, whereas in general the notion of s -concave measures makes sense for any $s \in [-\infty, \infty]$. We say that a measure μ on \mathbb{R}^n is s -concave if for any non-empty compact sets $A, B \subset \mathbb{R}^n$ we have

$$\mu(\lambda A + (1 - \lambda)B)^s \geq \lambda \mu(A)^s + (1 - \lambda) \mu(B)^s. \quad (3)$$

Similarly, a measure μ is called log-concave (or 0-concave) if for any compact sets $A, B \subset \mathbb{R}^n$ we have

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}. \quad (4)$$

We say that the support of measure μ is non-degenerate if it is not contained in any affine subspace of \mathbb{R}^n of dimension less than n . It was proved by Borell (see [2]) that a measure μ , with non-degenerate support, is log-concave if and only if it has a log-concave density, i.e. a density of the form $\varphi = e^{-V}$, where V is convex (and may attain value $+\infty$). Moreover, μ is s -concave with $s \in (0, 1/n)$ if and only if it has a density φ such that $\varphi^{\frac{s}{1-sn}}$ is concave. In the case $s = 1/n$ the density has to satisfy the strongest condition $\varphi(\lambda x + (1 - \lambda)y) \geq \max(\varphi(x), \varphi(y))$. An example of such measure is the uniform measure on a convex body $K \subset \mathbb{R}^n$. Let us also notice that a measure with non-degenerate support cannot be s -concave with $s > 1/n$. It can be seen by taking $\tilde{A} = \varepsilon A$ and $\tilde{B} = \varepsilon B$ in (3), sending $\varepsilon \rightarrow 0^+$ and comparing the limit with the Lebesgue measure.

Inequality (2) says that the Lebesgue measure is log-concave, whereas (1) means that it is also $1/n$ -concave. In general log-concavity does not imply s -concavity for $s > 0$. Indeed, consider the standard Gaussian measure γ_n on \mathbb{R}^n , i.e., the measure with density $(2\pi)^{-n/2} \exp(-|x|^2/2)$. This density is clearly log-concave and therefore γ_n satisfies (4). To see that γ_n does not satisfy (3) for $s > 0$ it suffices to take $B = \{x\}$ and send $x \rightarrow \infty$. Then the left hand side converges to 0 while the right hand side stays equal to $\lambda \mu(A)^s$, which is strictly positive for $\lambda > 0$ and $\mu(A) > 0$.

One might therefore ask whether (3) holds true for γ_n if we restrict ourselves to some special class of subsets of \mathbb{R}^n . In [13] R. Gardner and the fourth named author conjectured (Question 7.1) that

$$\gamma_n(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \gamma_n(A)^{1/n} + (1 - \lambda) \gamma_n(B)^{1/n} \quad (5)$$

holds true for any closed convex sets with $0 \in A \cap B$ and $\lambda \in [0, 1]$ and verified this conjecture in the following cases:

- (a) when A and B are products of intervals containing the origin,
- (b) when $A = [-a_1, a_2] \times \mathbb{R}^{n-1}$, where $a_1, a_2 > 0$ and B is arbitrary,
- (c) when $A = aK$ and $B = bK$ where $a, b > 0$ and K is a convex set, symmetric with respect to the origin.

It is interesting to note that the case (c) is related to the B-conjecture for Gaussian measures proposed by Banaszczyk (see [16]) and solved by Cordero-Erausquin, Fradelizi, and Maurey (see [7]). It states that for any convex symmetric set K the function $t \mapsto \gamma_n(e^t K)$ is log-concave. The B-conjecture is asking the same question for the general class of the even log-concave measures. It was shown in [7] that the conjecture is true for the case of unconditional log-concave measures and unconditional sets (see the definition below). Moreover, the conjecture has an affirmative answer for $n = 2$ due to the works of Livne Bar-on [17] and of Saroglou [28]. In [28] the proof is done by linking the problem to the new log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang, see [5], [6], [27] and [28]. In [22] the second named author proved that the assertion of the B-conjecture for a measure μ with a radially decreasing density and a symmetric convex body K formally implies the $1/n$ -concavity of the measure μ on the set of dilates of K .

In [23] T. Tkocz and the third named author showed that in general (5) is false under the assumption $0 \in A \cap B$. For sufficiently small $\varepsilon > 0$ and $\alpha < \pi/2$ sufficiently close to $\pi/2$ the pair of sets

$$A = \{(x, y) \in \mathbb{R}^2 : y \geq |x| \tan \alpha\}, \quad B = \{(x, y) \in \mathbb{R}^2 : y \geq |x| \tan \alpha - \varepsilon\}$$

serves as a counterexample. The authors however conjectured that (5) should be true for (centrally) symmetric convex bodies A, B .

One of the most important Brunn-Minkowski type inequalities for the Gaussian measure is Ehrhard's inequality, which states that for any two non-empty compact sets $A, B \subset \mathbb{R}^n$ and any $\lambda \in [0, 1]$ we have

$$\Phi^{-1}(\gamma_n(\lambda A + (1 - \lambda)B)) \geq \lambda \Phi^{-1}(\gamma_n(A)) + (1 - \lambda) \Phi^{-1}(\gamma_n(B)), \quad (6)$$

where $\Phi(t) = \gamma_1((-\infty, t])$. This inequality has been considered for the first time by Ehrhard in [9], where the author proved it assuming that both A and B are convex. Then Latała in [15] generalized Ehrhard's result to the case of arbitrary A and convex B . In its full generality, the inequality (6) has been established by Borell, [4] (see also [1]). Note that (5) is an inequality of the same type, with $\Phi(t)$ replaced with t^n , but none of them is a direct consequence of the other. The crucial property of Ehrhard's inequality is that it (in fact a more general form where λ and $1 - \lambda$ are replaced with α and β , under the conditions $\alpha + \beta \geq 1$ and $|\alpha - \beta| \leq 1$) gives the Gaussian isoperimetry as a simple consequence.

In this paper, \mathcal{K} denotes a family of sets closed under dilations, i.e., $A \in \mathcal{K}$ implies $tA \in \mathcal{K}$ for any $t \geq 0$. In particular, we assume that for any $A \in \mathcal{K}$ we have $0 \in A$. Classical families of such sets include the class of star-shaped bodies, the class of convex bodies containing the origin, the class of symmetric bodies and the class of unconditional bodies.

A general form of the Brunn-Minkowski inequality can be stated as follows.

Definition 1. We say that a Borel measure μ on \mathbb{R}^n satisfies the Brunn-Minkowski inequality in the class of sets \mathcal{K} if for any $A, B \in \mathcal{K}$ and for any $\lambda \in [0, 1]$ we have

$$\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \mu(A)^{1/n} + (1 - \lambda) \mu(B)^{1/n}. \quad (7)$$

Before we state our results, we introduce some basic notation and definitions.

Definition 2.

1. We say that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is unconditional if for any choice of signs $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ and any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we have $f(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) = f(x)$.
2. We say that an unconditional function is decreasing if for any $1 \leq i \leq n$ and any real numbers $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ the function

$$t \mapsto f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$$

is non-increasing on $[0, \infty)$.

3. A set $A \subseteq \mathbb{R}^n$ is called an *ideal* if $\mathbf{1}_A$ is unconditional and decreasing. In other words, a set $A \subset \mathbb{R}^n$ is an ideal if $(x_1, \dots, x_n) \in A$ implies $(\delta_1 x_1, \dots, \delta_n x_n) \in A$ for any choice of $\delta_1, \dots, \delta_n \in [-1, 1]$. The class of all ideals (in \mathbb{R}^n) will be denoted by \mathcal{K}_I .
4. A set $A \subseteq \mathbb{R}^n$ is called symmetric if $A = -A$. The class of all symmetric convex sets in \mathbb{R}^n will be denoted by \mathcal{K}_S .
5. A measure μ on \mathbb{R}^n is called unconditional if it has an unconditional density.

We note that the class of ideals contains the class of unconditional convex bodies, but it also contains some non-convex sets. For example, $B_p^n = \{x \in \mathbb{R}^n : \sum |x_i|^p \leq 1\}$ for $p \in (0, 1)$ are ideals. We also note that if an unconditional measure μ on \mathbb{R}^n is a product measure, i.e. $\mu = \mu_1 \otimes \dots \otimes \mu_n$, then the measures μ_i are even on \mathbb{R} .

Our first theorem reads as follows.

Theorem 1. Let μ be an unconditional product measure with decreasing density. Then μ satisfies the Brunn-Minkowski inequality in the class \mathcal{K}_I of all ideals in \mathbb{R}^n .

In addition, the Examples 1 and 2 at the end of the paper show that neither the assumption that μ is a product measure, nor the unconditionality of our sets A and B can be dropped.

In the second part of this article we provide a link between the Brunn-Minkowski inequality and the log-Brunn-Minkowski inequality. To state our observation we need two definitions.

Definition 3. Let \mathcal{K} be a class of subsets closed under dilations. We say that a family $\odot = (\odot_\lambda)_{\lambda \in [0, 1]}$ of functions $\mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ is a geometric mean if for any $A, B \in \mathcal{K}$ the set $A \odot_\lambda B$ is measurable, satisfies an inclusion $A \odot_\lambda B \subseteq \lambda A + (1 - \lambda)B$, and $(sA) \odot_\lambda (tB) = s^\lambda t^{1-\lambda} (A \odot_\lambda B)$, for any $s, t > 0$.

Definition 4. We say that a Borel measure μ on \mathbb{R}^n satisfies the log-Brunn-Minkowski inequality in the class of sets \mathcal{K} with a geometric mean \odot , if for any sets $A, B \in \mathcal{K}$ and for any $\lambda \in [0, 1]$ we have

$$\mu(A \odot_\lambda B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

Remark 1. We shall use two different geometric means. The first one is the geometric mean $\odot^S : \mathcal{K}_S \times \mathcal{K}_S \rightarrow \mathcal{K}_S$, defined by the formula

$$A \odot_\lambda^S B = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_A^\lambda(u) h_B^{1-\lambda}(u), \forall u \in S^{n-1}\}.$$

Here h_A is the support function of A , i.e., $h_A(u) = \sup_{x \in A} \langle x, u \rangle$ (see, [12], [29]).

The second mean $\odot^I : \mathcal{K}_I \times \mathcal{K}_I \rightarrow \mathcal{K}_I$ is defined by

$$A \odot_\lambda^I B = \bigcup_{x \in A, y \in B} [-|x_1|^\lambda |y_1|^{1-\lambda}, |x_1|^\lambda |y_1|^{1-\lambda}] \times \dots \times [-|x_n|^\lambda |y_n|^{1-\lambda}, |x_n|^\lambda |y_n|^{1-\lambda}].$$

It is straightforward to check, with the help of the inequality $a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b$, $a, b \geq 0$, that both means are indeed geometric.

In the Section 3 we prove the following proposition.

Proposition 1. Suppose that a Borel measure μ with a radially decreasing density f , i.e. density satisfying $f(tx) \geq f(x)$ for any $x \in \mathbb{R}^n$ and $t \in [0, 1]$, satisfies the log-Brunn-Minkowski inequality, with a geometric mean \odot , in a certain class of sets \mathcal{K} . Then μ satisfies the Brunn-Minkowski inequality in the class \mathcal{K} .

Böröczky, Lutwak, Yang and Zhang [5], proved the log-Brunn-Minkowski inequality for the Lebesgue measure and symmetric convex bodies on \mathbb{R}^2 equipped with geometric mean \odot^S . Saroglou [28], generalized the inequality to the case of measures with even log-concave densities on \mathbb{R}^2 (see Corollary 3.3 therein). Thus, as a consequence of Proposition 1 and Remark 1, we get the following theorem.

Theorem 2. Let μ be a measure on \mathbb{R}^2 with an even log-concave density. Then μ satisfies the Brunn-Minkowski inequality in the class \mathcal{K}_S of all symmetric convex sets in \mathbb{R}^2 .

Moreover, in [7] (Proposition 8, see also Proposition 4.2 in [27]) the authors proved the following fact.

Theorem 3. The log-Brunn-Minkowski inequality holds true with the geometric mean \odot^I for any measure with unconditional log-concave density in the class \mathcal{K}_I of all ideals in \mathbb{R}^n .

For the sake of completeness, we recall the argument in Section 3. As a consequence, applying our Proposition 1 together with Remark 1, we deduce:

Theorem 4. Let μ be an unconditional log-concave measure on \mathbb{R}^n . Then μ satisfies the Brunn-Minkowski inequality in the class \mathcal{K}_I of all ideals in \mathbb{R}^n .

The rest of this article is organized as follows. In the next section we present the proof of Theorem 1. In Section 3 we prove Proposition 1 and recall the proof of Theorem 3. In Section 4 we present applications of the above results. In the last section we discuss equality cases in Theorem 2 and Theorem 4. We also give examples showing optimality of Theorem 1 and state some open questions.

2 Proof of Theorem 1

Our strategy is to prove a certain functional version of (7). A functional version of the classical Brunn-Minkowski inequality is called the Prékopa-Leindler inequality, see [11] for the proof.

Prékopa-Leindler inequality, [26], [20]: Let f, g, m be non-negative measurable functions on \mathbb{R}^n and let $\lambda \in [0, 1]$. If for all $x, y \in \mathbb{R}^n$ we have $m(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$ then

$$\int m \, dx \geq \left(\int f \, dx \right)^\lambda \left(\int g \, dx \right)^{1-\lambda}.$$

Here we prove a version of the above inequality under the assumption of unconditionality of functions f, g and m .

Proposition 2. Fix $\lambda, p \in (0, 1)$. Suppose that m, f, g are unconditional decreasing non-negative functions and let μ be an unconditional product measure with decreasing density on \mathbb{R}^n . Assume that for any $x, y \in \mathbb{R}^n$ we have

$$m(\lambda x + (1 - \lambda)y) \geq f(x)^p g(y)^{1-p}.$$

Then

$$\int m \, d\mu \geq \left[\left(\frac{\lambda}{p} \right)^p \left(\frac{1-\lambda}{1-p} \right)^{1-p} \right]^n \left(\int f \, d\mu \right)^p \left(\int g \, d\mu \right)^{1-p}.$$

The above proposition allows us to prove the following lemma, which is in fact a reformulation of Theorem 1.

Lemma 1. Let A, B be ideals in \mathbb{R}^n and let μ be an unconditional product measure with decreasing density on \mathbb{R}^n . Then for any $\lambda \in [0, 1]$ and $p \in (0, 1)$ we have

$$\mu(\lambda A + (1 - \lambda)B) \geq \left[\left(\frac{\lambda}{p} \right)^p \left(\frac{1 - \lambda}{1 - p} \right)^{1-p} \right]^n \mu(A)^p \mu(B)^{1-p}.$$

It is worth noticing that the factor on the right hand side of this inequality replaces in some sense the lack of homogeneity of our measure μ . The main idea of the proof is to introduce an additional parameter $p \neq \lambda$ and do the optimization with respect to p .

We first show how Lemma 1 implies Theorem 1.

Proof of Theorem 1. Without loss of generality we assume that $\lambda \in (0, 1)$. Let us assume for a moment that $\mu(A)\mu(B) > 0$. Then we can use Lemma 1 with

$$p = \frac{\lambda\mu(A)^{1/n}}{\lambda\mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}} \in (0, 1). \quad (8)$$

Note that

$$\frac{\lambda}{p} = \frac{\lambda\mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}}{\mu(A)^{1/n}}, \quad \frac{1 - \lambda}{1 - p} = \frac{\lambda\mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}}{\mu(B)^{1/n}}.$$

Then

$$\left[\left(\frac{\lambda}{p} \right)^p \left(\frac{1 - \lambda}{1 - p} \right)^{1-p} \right]^n \mu(A)^p \mu(B)^{1-p} = \left(\lambda\mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n} \right)^n.$$

Thus the inequality in Lemma 1 becomes

$$\mu(\lambda A + (1 - \lambda)B) \geq \left(\lambda\mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n} \right)^n.$$

Now suppose that, say, $\mu(B) = 0$. Since B is a non-empty ideal, we have $0 \in B$. Therefore, $\lambda A \subseteq \lambda A + (1 - \lambda)B$. Let φ be the unconditional decreasing density of μ . Hence,

$$\begin{aligned} \mu(\lambda A + (1 - \lambda)B) &\geq \mu(\lambda A) = \int_{\lambda A} \varphi(x) \, dx = \lambda^n \int_A \varphi(\lambda y) \, dy \\ &= \lambda^n \int_A \varphi(\lambda y_1, \dots, \lambda y_n) \, dy = \lambda^n \int_A \varphi(\lambda|y_1|, \dots, \lambda|y_n|) \, dy \\ &\geq \lambda^n \int_A \varphi(|y_1|, \dots, |y_n|) \, dy = \lambda^n \mu(A). \end{aligned}$$

Therefore,

$$\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda\mu(A)^{1/n} = \lambda\mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}.$$

□

Next we show that Proposition 2 implies Lemma 1.

Proof of Lemma 1. We can assume that $\lambda \in (0, 1)$. Let us take $m(x) = \mathbf{1}_{\lambda A + (1 - \lambda)B}(x)$, $f(x) = \mathbf{1}_A(x)$, $g(x) = \mathbf{1}_B(x)$. Clearly, f, g and m are unconditional and decreasing, and verify $m(\lambda x + (1 - \lambda)y) \geq f(x)^p g(y)^{1-p}$ for any $p \in (0, 1)$. Our assertion follows from Proposition 2. □

For the proof of Proposition 2 we need a one dimensional Brunn-Minkowski inequality for unconditional measures.

Lemma 2. Let A, B be two symmetric intervals and let μ be an unconditional measure with decreasing density on \mathbb{R} . Then for any $\lambda \in [0, 1]$ we have

$$\mu(\lambda A + (1 - \lambda)B) \geq \lambda\mu(A) + (1 - \lambda)\mu(B).$$

Proof. We can assume that $A = [-a, a]$ and $B = [-b, b]$ for some $a, b > 0$. Let φ be the density of μ . Then our assertion is equivalent to

$$\int_0^{\lambda a + (1 - \lambda)b} \varphi(x) dx \geq \lambda \int_0^a \varphi(x) dx + (1 - \lambda) \int_0^b \varphi(x) dx.$$

In other words, the function $t \mapsto \int_0^t \varphi(x) dx$ should be concave on $[0, \infty)$. This is equivalent to $t \mapsto \varphi(t)$ being non-increasing on $[0, \infty)$. \square

Proof of Proposition 2. We proceed by induction on n . Let us begin with the case $n = 1$. We can assume that $\|f\|_\infty, \|g\|_\infty > 0$. If we multiply the functions m, f, g by positive numbers c_m, c_f, c_g satisfying $c_m = c_f^p c_g^{1-p}$, the hypothesis and the assertion do not change. Therefore, taking $c_f = \|f\|_\infty^{-1}$, $c_g = \|g\|_\infty^{-1}$, $c_m = \|f\|_\infty^{-p} \|g\|_\infty^{-(1-p)}$ we can assume that $\|f\|_\infty = \|g\|_\infty = 1$. Then the sets $\{f > t\}$ and $\{g > t\}$ are non-empty for $t \in (0, 1)$. Moreover, $\lambda\{f > t\} + (1 - \lambda)\{g > t\} \subseteq \{m > t\}$. Indeed, if $x \in \{f > t\}$ and $y \in \{g > t\}$ then $m(\lambda x + (1 - \lambda)y) \geq f(x)^p g(y)^{1-p} > t^p t^{1-p} = t$. Thus, $\lambda x + (1 - \lambda)y \in \{m > t\}$. Therefore, using Lemma 2, we get

$$\begin{aligned} \int m d\mu &= \int_0^\infty \mu(\{m > t\}) dt \geq \int_0^1 \mu(\lambda\{f > t\} + (1 - \lambda)\{g > t\}) dt \\ &\geq \lambda \int_0^1 \mu(\{f > t\}) dt + (1 - \lambda) \int_0^1 \mu(\{g > t\}) dt \\ &= \lambda \int f d\mu + (1 - \lambda) \int g d\mu. \end{aligned}$$

Now, using the inequality $pa + (1 - p)b \geq a^p b^{1-p}$, $a, b \geq 0$, we get

$$\lambda \int f d\mu + (1 - \lambda) \int g d\mu = p \frac{\lambda}{p} \int f d\mu + (1 - p) \frac{1 - \lambda}{1 - p} \int g d\mu \quad (9)$$

$$\geq \left(\frac{\lambda}{p}\right)^p \left(\frac{1 - \lambda}{1 - p}\right)^{1-p} \left(\int f d\mu\right)^p \left(\int g d\mu\right)^{1-p}. \quad (10)$$

Next, we do the induction step. Let us assume that the assertion is true in dimension $n - 1$. Let $m, f, g : \mathbb{R}^n \rightarrow [0, \infty)$ be unconditional decreasing. For $x_0, y_0, z_0 \in \mathbb{R}$ we define functions $m_{z_0}, f_{x_0}, g_{y_0}$ by

$$m_{z_0}(x) = m(z_0, x), \quad f_{x_0}(x) = f(x_0, x), \quad g_{y_0}(x) = g(y_0, x).$$

Clearly, these functions are also unconditional. Moreover, due to our assumptions on m, f, g we have

$$\begin{aligned} m_{\lambda x_0 + (1 - \lambda)y_0}(\lambda x + (1 - \lambda)y) &= m(\lambda x_0 + (1 - \lambda)y_0, \lambda x + (1 - \lambda)y) \\ &\geq f(x_0, x)^p g(y_0, y)^{1-p} = f_{x_0}(x)^p g_{y_0}(y)^{1-p}. \end{aligned}$$

Let us decompose μ in the form $\mu = \mu_1 \times \bar{\mu}$, where μ_1 is a measure on \mathbb{R} . Note that μ_1 and $\bar{\mu}$ are unconditional and $\bar{\mu}$ is a product measure on \mathbb{R}^{n-1} . Thus, by our induction assumption we have

$$\int m_{\lambda x_0 + (1 - \lambda)y_0} d\bar{\mu} \geq \left[\left(\frac{\lambda}{p}\right)^p \left(\frac{1 - \lambda}{1 - p}\right)^{1-p} \right]^{n-1} \left(\int f_{x_0} d\bar{\mu}\right)^p \left(\int g_{y_0} d\bar{\mu}\right)^{1-p}. \quad (11)$$

Now we define the functions

$$M(z_0) = \left[\left(\frac{\lambda}{p} \right)^p \left(\frac{1-\lambda}{1-p} \right)^{1-p} \right]^{-(n-1)} \int m_{z_0}(\xi) \, d\bar{\mu}(\xi), \quad (12)$$

$$F(x_0) = \int f_{x_0}(\xi) \, d\bar{\mu}(\xi), \quad G(y_0) = \int g_{y_0}(\xi) \, d\bar{\mu}(\xi). \quad (13)$$

Using inequality (11) we immediately get that

$$M(\lambda x_0 + (1-\lambda)y_0) \geq F(x_0)^p G(y_0)^{1-p}.$$

Moreover, it is easy to see that M, F, G are unconditional decreasing on \mathbb{R} . Thus, using Lemma 2 (the one-dimensional case), we get

$$\int M(z_0) \, d\mu_1(z_0) \geq \left(\frac{\lambda}{p} \right)^p \left(\frac{1-\lambda}{1-p} \right)^{1-p} \left(\int F(x_0) \, d\mu_1(x_0) \right)^p \left(\int G(y_0) \, d\mu_1(y_0) \right)^{1-p}. \quad (14)$$

Observe that

$$\begin{aligned} \int M(z_0) \, d\mu_1(z_0) &= \left[\left(\frac{\lambda}{p} \right)^p \left(\frac{1-\lambda}{1-p} \right)^{1-p} \right]^{-(n-1)} \int \int m_{z_0}(\xi) \, d\mu_{n-1}(\xi) \, d\mu_1(z_0) \\ &= \left[\left(\frac{\lambda}{p} \right)^p \left(\frac{1-\lambda}{1-p} \right)^{1-p} \right]^{-(n-1)} \int m \, d\mu. \end{aligned}$$

Similarly,

$$\int F(x_0) \, d\mu_1(x_0) = \int f \, d\mu, \quad \int G(y_0) \, d\mu_1(y_0) = \int g \, d\mu.$$

Our assertion follows. □

3 Proof of Proposition 1

In this section we first prove Proposition 1. The argument has a flavour of our previous proof.

Proof of Proposition 1. Let us first assume that $\mu(A)\mu(B) > 0$. From the definition of geometric mean we have $A \odot_p B \subseteq pA + (1-p)B$, for any $p \in (0, 1)$. Thus,

$$\begin{aligned} \mu(\lambda A + (1-\lambda)B) &= \mu \left(p \cdot \frac{\lambda}{p} A + (1-p) \cdot \frac{1-\lambda}{1-p} B \right) \geq \mu \left(\left(\frac{\lambda}{p} A \right) \odot_p \left(\frac{1-\lambda}{1-p} B \right) \right) \\ &= \mu \left(\left(\frac{\lambda}{p} \right)^p \left(\frac{1-\lambda}{1-p} \right)^{1-p} A \odot_p B \right). \end{aligned}$$

Let $t = \left(\frac{\lambda}{p} \right)^p \left(\frac{1-\lambda}{1-p} \right)^{1-p}$ and $C = A \odot_p B$. From the concavity of the logarithm it follows that $0 \leq t \leq 1$. We have

$$\mu(tC) = \int_{tC} f(x) \, dx = t^n \int_C f(tx) \, dx \geq t^n \int_C f(x) \, dx = t^n \mu(C). \quad (15)$$

Therefore,

$$\mu(\lambda A + (1 - \lambda)B) \geq t^n \mu(A \odot_p B) \geq t^n \mu(A)^p \mu(B)^{1-p} = \left[\left(\frac{\lambda}{p} \right)^p \left(\frac{1 - \lambda}{1 - p} \right)^{1-p} \right]^n \mu(A)^p \mu(B)^{1-p}.$$

Taking

$$p = \frac{\lambda \mu(A)^{1/n}}{\lambda \mu(A)^{1/n} + (1 - \lambda) \mu(B)^{1/n}} \quad (16)$$

gives

$$\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \mu(A)^{1/n} + (1 - \lambda) \mu(B)^{1/n}.$$

If, say, $\mu(B) = 0$ then by (15), applied for C replaced with A , and the fact that $0 \in B$ we get

$$\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \mu(\lambda A)^{1/n} \geq \lambda \mu(A)^{1/n} = \lambda \mu(A)^{1/n} + (1 - \lambda) \mu(B)^{1/n}.$$

□

We now sketch the proof of Theorem 3.

Proof. Let $A, B \in \mathcal{K}_I$ and let us take $f, g, m : [0, +\infty)^n \rightarrow [0, +\infty)$ given by $f = \mathbf{1}_{A \cap [0, +\infty)^n}$, $g = \mathbf{1}_{B \cap [0, +\infty)^n}$ and $m = \mathbf{1}_{(A \odot_\lambda^I B) \cap [0, +\infty)^n}$. Let φ be the unconditional log-concave density of μ . We define

$$F(x) = f(e^{x_1}, \dots, e^{x_n}) \varphi(e^{x_1}, \dots, e^{x_n}) e^{x_1 + \dots + x_n}, \quad G(x) = g(e^{x_1}, \dots, e^{x_n}) \varphi(e^{x_1}, \dots, e^{x_n}) e^{x_1 + \dots + x_n},$$

$$M(x) = m(e^{x_1}, \dots, e^{x_n}) \varphi(e^{x_1}, \dots, e^{x_n}) e^{x_1 + \dots + x_n}.$$

One can easily check, using the definition of \mathcal{K}_I and the definition of the geometric mean \odot_λ^I , as well as the inequalities

$$\begin{aligned} & \varphi(e^{\lambda x_1 + (1-\lambda)y_1}, \dots, e^{\lambda x_n + (1-\lambda)y_n}) \\ & \geq \varphi(\lambda e^{x_1} + (1 - \lambda)e^{y_1}, \dots, \lambda e^{x_n} + (1 - \lambda)e^{y_n}) \geq \varphi(e^{x_1}, \dots, e^{x_n})^\lambda \varphi(e^{y_1}, \dots, e^{y_n})^{1-\lambda}, \end{aligned}$$

that the functions F, G, M satisfy the assumptions of the Prékopa-Leindler inequality. As a consequence, we get $\mu((A \odot_\lambda^I B) \cap [0, +\infty)^n) \geq \mu(A \cap [0, +\infty)^n)^\lambda \mu(B \cap [0, +\infty)^n)^{1-\lambda}$. The assertion follows from unconditionality of our measure μ and the fact that A, B and $A \odot_\lambda^I B$ are ideals. □

4 Applications

Let us describe some corollaries of the Brunn-Minkowski type inequality we established, which are analogues to well-known offsprings of the Brunn-Minkowski inequality for the volume. In what follows a pair (\mathcal{K}, μ) is called *nice* if one of the following three cases holds.

- (a) $\mathcal{K} = \mathcal{K}_I$ and μ is an unconditional, product measure with decreasing density on \mathbb{R}^n ,
- (b) $\mathcal{K} = \mathcal{K}_I$ and μ is an unconditional log-concave measure on \mathbb{R}^n ,
- (c) $\mathcal{K} = \mathcal{K}_S$ and μ is an even log-concave measure on \mathbb{R}^2 .

Corollary 1. Suppose that a pair (\mathcal{K}, μ) is nice. Let $A, B \subset \mathcal{K}$ be convex. Then the function $t \mapsto \mu(A + tB)^{1/n}$ is concave on $[0, \infty)$.

Indeed, for any $\lambda \in [0, 1]$ and $t_1, t_2 \geq 0$ we have

$$\begin{aligned} \mu(A + (\lambda t_1 + (1 - \lambda)t_2)B)^{1/n} &= \mu(\lambda(A + t_1B) + (1 - \lambda)(A + t_2B))^{1/n} \\ &\geq \lambda\mu(A + t_1B)^{1/n} + (1 - \lambda)\mu(A + t_2B)^{1/n}. \end{aligned}$$

Note that in the first line we have used the convexity of A and B . If $B = B_2^n$ is the unit Euclidean ball, the expression $\mu(A + tB)$ is called the parallel volume and has been studied in the case of the Lebesgue measure by Costa and Cover in [8] as an analogue of concavity of entropy power in Information theory. The authors conjectured that for any measurable set A the parallel volume is $1/n$ -concave. In [10], M. Fradelizi and the second named author proved that this conjecture is true for any measurable set in dimension 1 and for any connected set in dimension 2. However, the authors proved that this conjecture fails for arbitrary sets in dimension $n \geq 2$. In a recent paper [21] the second named author investigated the parallel volume $\mu(A + tB_2^n)$ in the context of s -concave measures as well as functional versions. Our Corollary 1 gives the Costa-Cover conjecture for any convex set $A \in \mathcal{K}$, where (\mathcal{K}, μ) is a nice pair. Moreover, B_2^n can be replaced with any convex set $B \in \mathcal{K}$.

Second, we state the following analogue of Brunn's theorem on volumes of sections of convex bodies (see [11], [12] and [29] for the volume case).

Corollary 2. Suppose that a pair (\mathcal{K}, μ) is nice. Let $A \in \mathcal{K}$ be a convex set and let φ be the density of μ . Then the function $t \mapsto \mu_{n-1}(A \cap \{x_1 = t\})$ is $\frac{1}{n-1}$ -concave on its support, where

$$\mu_{n-1}(A \cap \{x_1 = t\}) = \int_{(t, x_2, \dots, x_n) \in A} \varphi(t, x_2, \dots, x_n) dx_2 \dots dx_n.$$

Indeed, let us denote $A_{\{x_1=t\}} = A \cap \{x_1 = t\}$. By convexity of A we get

$$\lambda A_{\{x_1=t_1\}} + (1 - \lambda)A_{\{x_1=t_2\}} \subseteq A_{\{x_1=\lambda t_1+(1-\lambda)t_2\}}.$$

Thus, using (7), for any $\lambda \in [0, 1]$ and $t_1, t_2 \in \mathbb{R}$ such that $A_{\{x_1=t_1\}}$ and $A_{\{x_1=t_2\}}$ are both non-empty, we get

$$\begin{aligned} \mu_{n-1}(A_{\{x_1=\lambda t_1+(1-\lambda)t_2\}})^{\frac{1}{n-1}} &\geq \mu_{n-1}(\lambda A_{\{x_1=t_1\}} + (1 - \lambda)A_{\{x_1=t_2\}})^{\frac{1}{n-1}} \\ &\geq \lambda\mu_{n-1}(A_{\{x_1=t_1\}})^{\frac{1}{n-1}} + (1 - \lambda)\mu_{n-1}(A_{\{x_1=t_2\}})^{\frac{1}{n-1}}. \end{aligned}$$

Third, let us mention the relation of our result to the Gaussian isoperimetric inequality and the S-inequality. The Gaussian isoperimetric inequality (established by Sudakov and Tsirelson, [30], and independently by Borell, [3]), states that for any measurable set $A \subset \mathbb{R}^n$ and any $t > 0$, the quantity $\gamma_n(A_t)$ is minimized, among all sets with prescribed measure, for the half spaces $H_{a,\theta} = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \leq a\}$, with $a \in \mathbb{R}$ and $\theta \in S^{n-1}$. Infinitesimally, it says that among all sets with prescribed measure the half spaces are those with the smallest Gaussian surface area, i.e., the quantity

$$\gamma_n^+(\partial A) = \liminf_{t \rightarrow 0^+} \frac{\gamma_n(A + tB_2^n) - \gamma_n(A)}{t}.$$

The S-inequality of Latała and Oleszkiewicz, see [18], states that for any $t > 1$ and any symmetric convex body A the quantity $\gamma_n(tA)$ is minimized, among all subsets with prescribed measure, for the strip of the form $S_L = \{x \in \mathbb{R}^n : |x_1| \leq L\}$. This result admits an equivalent infinitesimal version, namely, among all symmetric convex bodies A with prescribed Gaussian measure the strip S_L minimizes the quantity $\frac{d}{dt}\gamma_n(tA)|_{t=1}$, which is equivalent to maximizing

$$M_{\gamma_n}(A) = \int_A |x|^2 d\gamma_n(x),$$

see [14] or [25]. For a general measure μ with a density $e^{-\psi}$, one can show that the infinitesimal version of S-inequality is an issue of maximizing the quantity

$$M_\mu(A) = \int_A \langle x, \nabla \psi(x) \rangle d\mu(x), \quad (17)$$

see equation (22) below. Not much is known about an analogue of S -inequality in the case of general measure. In the unconditional case it has been solved for some particular product measures like products of Gamma and Weibull distributions, see [24]. It turns out that inequality (5) implies a certain mixture of Gaussian isoperimetry and reverse S-inequality. Namely, we have the following corollary.

Corollary 3. Let A be an ideal in \mathbb{R}^n (or a general symmetric convex set in \mathbb{R}^2) and let $r > 0$. Then we have

$$r\gamma_n^+(\partial A) + M_{\gamma_n}(A) \geq n\gamma_n(rB_2^n)^{\frac{1}{n}} \gamma_n(A)^{1-\frac{1}{n}}$$

with equality for $A = rB_2^n$.

Let us note that

$$\begin{aligned} \gamma_n(rB_2^n + \varepsilon B_2^n) &= (2\pi)^{-n/2} (r + \varepsilon)^n \int_{B_2^n} e^{-\frac{|(r+\varepsilon)x|^2}{2}} dx \\ &= (2\pi)^{-n/2} (r^n + nr^{n-1}\varepsilon + o(\varepsilon)) \int_{B_2^n} e^{-\frac{|rx|^2}{2}} (1 - \varepsilon r|x|^2 + o(\varepsilon)) dx \\ &= \gamma_n(rB_2^n) + \frac{\varepsilon}{r} (n\gamma_n(rB_2^n) - M_{\gamma_n}(rB_2^n)) + o(\varepsilon). \end{aligned}$$

Thus,

$$r\gamma_n^+(\partial(rB_2^n)) = n\gamma_n(rB_2^n) - M_{\gamma_n}(rB_2^n).$$

Hence, if $\gamma_n(A) = \gamma_n(rB_2^n)$ in Corollary 3, then we get

$$r\gamma_n^+(\partial A) + M_{\gamma_n}(A) \geq r\gamma_n^+(\partial(rB_2^n)) + M_{\gamma_n}(rB_2^n). \quad (18)$$

In other words, Euclidean balls minimize the quantity $r\gamma_n^+(\partial A) + M_{\gamma_n}(A)$ among ideals in \mathbb{R}^n (or symmetric convex sets in \mathbb{R}^2) with prescribed measure.

It is known that among all symmetric convex sets (in fact among all measurable sets) with prescribed Gaussian measure, the quantity $M_{\gamma_n}(A)$ is minimized by Euclidean balls rB_2^n (this fact can be seen as a reverse S-inequality). Indeed, suppose that $\gamma_n(A) = \gamma_n(rB_2^n)$. Then

$$\begin{aligned} M_{\gamma_n}(A) - M_{\gamma_n}(rB_2^n) &= \int_{A \setminus (rB_2^n)} |x|^2 d\gamma_n(x) - \int_{(rB_2^n) \setminus A} |x|^2 d\gamma_n(x) \\ &\geq r^2(\gamma_n(A \setminus (rB_2^n)) - \gamma_n((rB_2^n) \setminus A)) = 0. \end{aligned}$$

However, in general the quantity $\gamma_n^+(\partial A)$ is not minimized by Euclidean balls, e.g., one can check that for large values of $\gamma_2(A)$ the symmetric strip has smaller Gaussian surface area than the Euclidean ball, see [19, Lemma 3]. Hence, inequality (18) is a new isoperimetric-type inequality that links the Gaussian isoperimetry and reverse S-inequality.

Let us state and prove a more general version of Corollary 3. Let $\mu^+(\partial A)$ be the μ surface area of A , i.e.,

$$\mu^+(\partial A) = \liminf_{t \rightarrow 0^+} \frac{\mu(A + tB_2^n) - \mu(A)}{t}.$$

Let

$$V_1^\mu(A, B) = \frac{1}{n} \liminf_{t \rightarrow 0^+} \frac{\mu(A + tB) - \mu(A)}{t}$$

be the first mixed volume of arbitrary sets A and B , with respect to measure μ . Clearly, $\mu^+(\partial A) = nV_1^\mu(A, B_2^n)$.

Corollary 4. Let $A, B \in \mathcal{K}$ and suppose that (\mathcal{K}, μ) is a nice pair. Then we have

$$V_1^\mu(A, B) + \frac{1}{n}M_\mu(A) \geq \mu(B)^{1/n}\mu(A)^{1-1/n}. \quad (19)$$

In particular,

$$r\mu^+(\partial A) + M_\mu(A) \geq n\mu(rB_2^n)^{1/n}\mu(A)^{1-1/n}. \quad (20)$$

To prove this we note that for any sets $A, B \in \mathcal{K}$ and any $\varepsilon \in [0, 1)$ we have

$$\mu(A + \varepsilon B)^{1/n} \geq (1 - \varepsilon)\mu\left(\frac{A}{1 - \varepsilon}\right)^{1/n} + \varepsilon\mu(B)^{1/n}. \quad (21)$$

Indeed, it suffices to use Theorem 1 with $\lambda = 1 - \varepsilon$ and $\tilde{A} = A/(1 - \varepsilon)$, $\tilde{B} = B$. Note that for $\varepsilon = 0$ we have equality. Thus, differentiating (21) at $\varepsilon = 0$ we get

$$\frac{1}{n}\mu(A)^{\frac{1}{n}-1} \cdot nV_1^\mu(A, B) \geq \mu(B)^{\frac{1}{n}} - \mu(A)^{\frac{1}{n}} + \frac{1}{n}\mu(A)^{\frac{1}{n}-1} \frac{d}{dt}\mu(tA)\Big|_{t=1}.$$

By changing variables we obtain

$$\frac{d}{dt}\mu(tA)\Big|_{t=1} = \frac{d}{dt} \int_A e^{-\psi(tx)} t^n dx \Big|_{t=1} = n\mu(A) - \int_A \langle x, \nabla\psi(x) \rangle d\mu(x) = n\mu(A) - M_\mu(A). \quad (22)$$

Thus,

$$\mu(A)^{\frac{1}{n}-1}V_1^\mu(A, B) \geq \mu(B)^{\frac{1}{n}} - \frac{1}{n}\mu(A)^{\frac{1}{n}-1}M_\mu(A),$$

which is exactly (19). To get (20) one has to take $B = rB_2^n$ in (19).

The above inequalities can be seen as an analogue of the so-called Minkowski first inequality for the Lebesgue measure (see [11], [12] and [29]), which says that for any two convex bodies A, B in \mathbb{R}^n we have

$$V_1^{\text{vol}_n}(A, B) \geq \text{vol}_n(A)^{1-\frac{1}{n}} \text{vol}_n(B)^{\frac{1}{n}}.$$

5 Examples and open problems

We first discuss equality cases in Theorem 2 and Theorem 4.

Remark 2. The equality in Theorem 2 and Theorem 4 is achieved only if A is a dilation of B . Indeed, in the proof of Proposition 1 we use the inclusion $\tilde{A} \odot_p \tilde{B} \subseteq p\tilde{A} + (1 - p)\tilde{B}$, where $\tilde{A} = \frac{\lambda}{p}A$ and $\tilde{B} = \frac{1-\lambda}{1-p}B$, with p given by (16). To have equality in (7) we need to have, in particular, equality in the above inclusion (with this particular choice of p). Notice that $a^pb^{1-p} = pa + (1 - p)b$, $a, b \geq 0$, if and only if $a = b$. Thus, $\tilde{A} \odot_p^S \tilde{B} = p\tilde{A} + (1 - p)\tilde{B}$ if and only if $\tilde{A} = \tilde{B}$ (by using the fact that $h_{\tilde{A}} = h_{\tilde{B}}$ if and only if $\tilde{A} = \tilde{B}$). Similarly, one has $\tilde{A} \odot_p^I \tilde{B} = p\tilde{A} + (1 - p)\tilde{B}$ if and only if $\tilde{A} = \tilde{B}$. This means that A is a dilation of B .

In general one cannot hope to have equality cases only if $A = B$. Let us illustrate this in the case of the Lebesgue measure. Indeed, then we have equality in (7) if $A = aK$ and $B = bK$, where K is some fixed convex set. In this case the equality $\tilde{A} = \tilde{B}$ leads to the condition $\frac{\lambda}{p}a = \frac{1-\lambda}{1-p}b$, which is equivalent to choosing $p = \frac{\lambda a}{\lambda a + (1-\lambda)b}$. This coincides with (16).

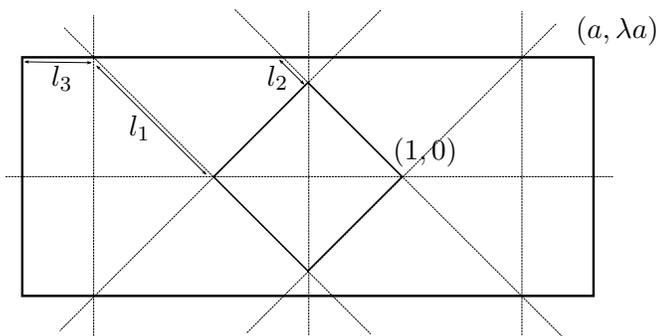
However, one can get $A = B$ as the only case of equality if one assumes that the density of μ is strictly decreasing. To see this it suffices to observe that for the equality in (7) we have to have $t = 1$ in the proof of Proposition 1, which leads to $\mu(A) = \mu(B)$. Together with the fact that A is a dilation of B we get $A = B$.

We also show that the assumptions of Theorem 1 are necessary. Namely, as long as we work with decreasing densities, which may not be log-concave, one has to assume that the measure is product and the sets are unconditional.

Example 1. The assumption, that our measure μ in Theorem 1 is a product, is important. Indeed, let us take the square $C = \{|x|, |y| \leq 1\} \subset \mathbb{R}^2$ and take the measure with density $\varphi(x) = \frac{1}{2}\mathbf{1}_{2C}(x) + \frac{1}{2}\mathbf{1}_C(x)$. This density is unconditional, however it is not a product. Let us define $\psi(a) = \sqrt{\mu(aC)}$. The assertion of Theorem 1 implies that ψ is concave. However, we have $\psi(a) = \sqrt{2a^2 + 2}$ for $a \in [1, 2]$, which is strictly convex. Thus, μ does not satisfy (7).

Example 2. In general, under the assumption that our measure μ is unconditional and a product, one cannot prove that Theorem 1 holds true for arbitrary symmetric convex sets. To see this, let us take the product measure $\mu = \mu_0 \otimes \mu_0$ on \mathbb{R}^2 , where μ_0 has an unconditional density $\varphi(x) = p + (1-p)\mathbf{1}_{[-1/\sqrt{2}, 1/\sqrt{2}]}(x)$ for some $p \in [0, 1]$.

To simplify the computation let us rotate the whole picture by angle $\pi/4$. Then consider the rectangle $R = [-1, 1] \times [-\lambda, \lambda]$ for $0 < \lambda \leq 1/2$. As in the previous example, it is enough to show that the function $\psi(a) = \sqrt{\mu(aR)}$ is not concave. Let us consider this function only on the interval $[1/\lambda, \infty)$. The condition $\lambda \leq 1/2$ ensures that the point $(a, \lambda a)$ lies in the region with density p^2 . Let us introduce lengths l_1, l_2, l_3 (see the picture below).



Note that $l_1 = \sqrt{2}\lambda a$, $l_2 = \sqrt{2}(\lambda a - 1)$ and $l_3 = a - (1 + \lambda a)$. Let $\omega(a) = \mu(aR)$. We have

$$\begin{aligned} \omega(a) &= 2 + 4\sqrt{2}p \cdot \frac{l_1 + l_2}{2} + p^2 l_1^2 + p^2 l_2^2 + 4p^2 l_3 \lambda a \\ &= 2 + 4p(2\lambda a - 1) + 2p^2 \lambda^2 a^2 + 2p^2 (\lambda a - 1)^2 + 4p^2 \lambda a (a - 1 - \lambda a) \\ &= 2(1-p)^2 + 4p\lambda a(pa + 2 - 2p) = d_0 + d_1 a + d_2 a^2, \end{aligned}$$

where $d_0 = 2(1-p)^2$, $d_1 = 8p(1-p)\lambda$, $d_2 = 4p^2\lambda$. We show that ψ is strictly convex for $p \in (0, 1)$ and $0 < \lambda < 1/2$. Indeed, $\psi'' > 0$ is equivalent to $2\omega\omega'' > (\omega')^2$. But

$$\begin{aligned} 2\omega(a)\omega''(a) - (\omega'(a))^2 &= 4d_2(d_0 + d_1 a + d_2 a^2) - (2d_2 a + d_1)^2 = 4d_2 d_0 - d_1^2 \\ &= 32\lambda p^2(1-p)^2 - 64\lambda^2 p^2(1-p)^2 = 32\lambda p^2(1-p)^2(1-2\lambda) > 0. \end{aligned}$$

We would like to finish the paper with a list of open questions that arose during our study.

Question. Let us assume that the measure μ has an even log-concave density (not-necessarily product).

- Does the assertion of Theorem 1 holds true for arbitrary symmetric sets A and B ?
- If not, is it true under additional assumption that the measure is product?
- In particular, can one remove the assumption of unconditionality in the Gaussian Brunn-Minkowski inequality?

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References

- [1] F. Barthe, N. Huet, *On Gaussian Brunn-Minkowski inequalities*, *Studia Math.* 191 (2009), 283–304.
- [2] C. Borell, *Convex set functions in d -space*, *Period. Math. Hungarica* 6 (1975), 111–136.
- [3] C. Borell, *The Brunn-Minkowski inequality in Gauss space*, *Invent. Math.* 30, 2 (1975), 207–216.
- [4] C. Borell, *The Ehrhard inequality*, *C. R. Math. Acad. Sci. Paris* 337 (2003), no. 10, 663–666.
- [5] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, *The log-Brunn-Minkowski inequality*, *Adv. Math.* 231 (2012), 1974–1997.
- [6] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, *The logarithmic Minkowski problem*, *J. Amer. Math. Soc.* 26 (2013), 831–852.
- [7] D. Cordero-Erausquin, M. Fradelizi, B. Maurey, *The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems*, *J. Funct. Anal.* 214 (2004), no. 2, 410–427.
- [8] M. Costa, T. M. Cover, *On the similarity of the entropy power inequality and the Brunn-Minkowski inequality*, *IEEE Trans. Inform. Theory* 30 (1984), no. 6, 837–839.
- [9] A. Ehrhard, *Symétrisation dans l'espace de Gauss*, *Math. Scand.* 53 (1983), 281–301.
- [10] M. Fradelizi, A. Marsiglietti, *On the analogue of the concavity of entropy power in the Brunn-Minkowski theory*, *Adv. in Appl. Math.* 57 (2014), 1–20.
- [11] R. J. Gardner, *The Brunn-Minkowski inequality*, *Bull. Amer. Math. Soc. (N.S.)* 39 (2002), no. 3, 355–405.
- [12] R. J. Gardner, *Geometric tomography*. Second edition. *Encyclopedia of Mathematics and its Applications*, 58 Cambridge University Press, Cambridge, 2006.
- [13] R. J. Gardner, A. Zvavitch, *Gaussian Brunn-Minkowski inequalities*, *Trans. Amer. Math. Soc.* 362 (2010), no. 10, 5333–5353.
- [14] S. Kwapien, J. Sawa, *On some conjecture concerning Gaussian measures of dilatations of convex symmetric sets*, *Studia Math.* 105 (1993), no. 2, 173–187.
- [15] R. Latała, *A note on the Ehrhard inequality*, *Studia Math.* 118 (1996) 169–174.
- [16] R. Latała, *On some inequalities for Gaussian measures*, *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, 813–822, Higher Ed. Press, Beijing, 2002.
- [17] A. Livne Bar-on, *The (B) conjecture for uniform measures in the plane*, *Geometric Aspects of Functional Analysis, Lecture Notes in Mathematics Volume 2116*, 2014, 341–353.
- [18] R. Latała, K. Oleszkiewicz, *Gaussian measures of dilatations of convex symmetric sets*, *Ann. Probab.* 27 (1999), 1922–1938.

- [19] R. Latała and K. Oleszkiewicz, *Small ball probability estimates in terms of widths*, *Studia Math.* 169 (2005), 305–314.
- [20] L. Leindler, *On a certain converse of Hölder’s inequality*, II, *Acta Sci. Math. (Szeged)* 33 (1972), 217–223.
- [21] A. Marsiglietti, *Concavity properties of extensions of the parallel volume*, *Mathematika* (2015), Available on CJO2015 doi:10.1112/S0025579314000369.
- [22] A. Marsiglietti, *On the improvement of concavity of convex measures*, *Proc. Amer. Math. Soc.* (2015), to appear, arXiv:1403.7643.
- [23] P. Nayar, T. Tkocz, *A note on a Brunn-Minkowski inequality for the Gaussian measure*, *Proc. Amer. Math. Soc.* 141 (2013), no. 11, 4027–4030.
- [24] P. Nayar, T. Tkocz, *S-inequality for certain product measures*, *Math. Nachr.* 287 (2014), no. 4, 398–404.
- [25] P. Nayar, T. Tkocz, *The unconditional case of the complex S-inequality*, *Israel J. Math.* 197 (2013), no. 1, 99–106.
- [26] A. Prékopa, *On logarithmic concave measures and functions*, *Acta Sci. Math.* 34 (1973), 335–343.
- [27] C. Saroglou, *Remarks on the conjectured log-Brunn-Minkowski inequality*, 2014, *Geom. Dedicata* (to appear), arXiv:1311.4954.
- [28] C. Saroglou, *More on logarithmic sums of convex bodies*, arXiv:1409.4346.
- [29] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition. *Encyclopedia of Mathematics and its Applications*, 151. Cambridge University Press, Cambridge, 2014.
- [30] V. N. Sudakov, B. S. Cirel’son, *Extremal properties of half-spaces for spherically invariant measures*, *Problems in the theory of probability distributions*, II. *Zap. Nauch. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 41 (1974), 165, 14–24.

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