Analytic structures in maximal ideal spaces

Manuel Maestre

Infinite Dimensional Analysis, October 28-30, 2016 Kent State University

RICHARD’S PARTY
Introduction. The algebras of holomorphic functions in $\mathbb{C}^N$.

Infinite dimensional setting. Size of the fibres

Injecting analytic structures

**Definition**

Given $U$ an open subset of $\mathbb{C}^n$, we will denote by

$$H(U)$$

the space of all functions $f : U \to \mathbb{C}$ which are holomorphic on $U$. It is a Fréchet algebra endowed with uniform convergence on compact subsets of $U$. 

For any open subset of a complex Banach space $X$, $H_\infty(U)$ is the Banach algebra of all functions $f : U \to \mathbb{C}$ which are holomorphic=Fréchet differentiable and bounded on $U$ with the supremum norm.

Given $x \in U$ there exists $L : X \to \mathbb{C}$ linear AND CONTINUOUS such that

$$\lim_{h \to 0} f(x + h) - f(x) - L(h) \parallel h \parallel = 0$$

In particular, $X^* \subset H(U)$. 

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For $U$ any open subset of a complex Banach space $X$, $H_\infty(U)$ is the Banach algebra of all functions $f : U \to \mathbb{C}$ which are holomorphic=Fréchet differentiable and bounded on $U$ with the supremum norm.
Introduction. The algebras of holomorphic functions in \( \mathbb{C}^N \).

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Ball Algebra

Given a complex Banach space $X$, and its open unit ball $B_X$, we will denote by $A_u(B_X)$ the Banach algebra of all functions $f : \bar{B}_X \to \mathbb{C}$ such that are uniformly continuous on $\bar{B}_X$ and holomorphic=Fréchet differentiable on $B_X$. 

Maximal ideal space

For $A_u$ either $H_{\infty}(U)$ or $A_u(B_X)$ the maximal ideal space (spectrum) $M(A)$ is the compact set of all non-null linear and multiplicative $\phi : A \to \mathbb{C}$ endowed with the weak-star topology $w(A^*, A)$. 

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For $A$ either $H_\infty(U)$ or $A_u(B_X)$ the maximal ideal space (spectrum) $\mathcal{M}(A)$ is the compact set of all non-null linear and multiplicative $\varphi : A \to \mathbb{C}$ endowed with the weak-star topology $w(A^* , A)$.
Maximal ideal space

The maximal ideal space (spectrum) \( M(H(U)) \) is set of all non-null linear and multiplicative \( \varphi : H(U) \to \mathbb{C} \) endowed with the weak-star topology \( w(H(U)^*, H(U)) \).
Maximal ideal space

The *maximal ideal space (spectrum)* $\mathcal{M}(H(U))$ is set of all non-null linear and multiplicative $\varphi : H(U) \rightarrow \mathbb{C}$ endowed with the weak-star topology $w(H(U)^*, H(U))$.

Remark

For any $U$ open subset of $\mathbb{C}$

$$\mathcal{M}(H(U)) = \{\delta_z : z \in U\}$$
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If not, then $\frac{1}{z-a} \in H(U)$,
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\mathcal{M}(H(U)) = \{ \delta_z : z \in U \}
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Let \( \phi \in \mathcal{M}(H(U)) \) and

\[
a = \phi(z \mapsto z).
\]

\( a \in U \)

If not, then \( \frac{1}{z-a} \in H(U) \),

\[
1 = \phi(1) = \phi\left( \frac{z-a}{z-a} \right) = \phi\left( \frac{1}{z-a} \right) \phi(z-a) = \phi\left( \frac{1}{z-a} \right)(\phi(z)-a) = 0.
\]
$\mathcal{M}(A_u(D)) = \{\delta_x : x \in \overline{D}\}$
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**Corona Theorem (Carleson, 1962)**

If we denote by \( \mathbb{D} \) the open unit disk of \( \mathbb{C} \)

\[ \mathcal{M}(H_\infty(\mathbb{D})) = \{ \delta_x : x \in \mathbb{D} \}^{w*}. \]
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\[ \mathcal{M}(A_u(D)) = \{ \delta_x : x \in \overline{D} \} \]

**Corona Theorem (Carleson, 1962)**

If we denote by $D$ the open unit disk of $\mathbb{C}$

\[ \mathcal{M}(H_\infty(D)) = \overline{\{ \delta_x : x \in D \}^{w*}}. \]

But!

\[ \beta^N \setminus N \subset \mathcal{M}(H_\infty(D)) \]
From now on $X$ is an infinite dimensional complex Banach space!!!

\[ \{ \delta_x : x \in B_X \} \subset \mathcal{M}(H_\infty(B_X)) \]

\[ \{ \delta_x : x \in \bar{B}_X \} \subset \mathcal{M}(A_u(B_X)) \]
A HAHN-BANACH EXTENSION THEOREM
FOR ANALYTIC MAPPINGS

by

RICHARD M. ARON (*) and PAUL. D. BERNER (\textsuperscript{2})

[Dublin]

Abstract. — Let $E$ be a closed subspace of a Banach space $G$, let $U$ be an open subset of $E$, and let $F$ be another Banach space. The problem of extending analytic $F$-valued mappings defined on $U$ to an open subset of $G$ is discussed, and necessary and sufficient conditions are found for such extensions to exist. These conditions involve the existence of a continuous linear extension mapping of $E'$ to $G'$, which in turn is related to the Hahn-Banach theorem for linear transformations.

We consider the problem of extending an analytic mapping defined on an open subset $U$ of a closed subspace $E$ of a Banach space $G$ to an analytic mapping defined on an open neighbourhood of $U$ in $G$. Our general approach is to obtain extensions to the whole space $G$ of polynomials defined on $E$, and then to use local Taylor series representations to extend analytic functions locally. It is necessary to show that the local extensions are “coherent in the overlaps”. This can be done when one can define a linear and continuous extension mapping taking polynomials defined on $E$ to their extensions defined on $G$, which in turn is closely related to the vector-valued Hahn-Banach property as studied by Nachbin, Lindenstrauss, and others.

The general question of extending analytic mappings on topological vector spaces was raised by Dineen in [4]. He and other authors (Hirschowitz,

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\[ \tilde{\delta} z : z \in B_*^X \subset M(\mathbb{H}_\infty(B_X)) \]

\[ \tilde{\delta} z : z \in \bar{B}_X^{**} \subset M(A_{\text{u}}(B_X)) \]

\[ \tilde{\delta} z < \{ \delta x : x \in B_X \} \] if \( z \in B_X^{**} \setminus B_X \).

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Theorem: Davie-Gamelin (after Aron-Berner)

\[
\{ \bar{\delta}_z : z \in B_{X^{**}} \} \subset \mathcal{M}(H_{\infty}(B_X)).
\]

\[
\{ \tilde{\delta}_z : z \in \overline{B}_{X^{**}} \} \subset \mathcal{M}(A_u(B_X)).
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\{ \tilde{\delta}_z : z \in \bar{B}_{X^{**}} \} \subset \mathcal{M}(A_u(B_X)).
\]

\[\tilde{\delta}_z \notin \{ \delta_x : x \in B_X \}\]

if \( z \in B_{X^{**}} \setminus B_X \).
Spectra of algebras of analytic functions on a Banach space

By R. M. Aron at Kent, B. J. Cole*) at Providence and T. W. Gamelin**) at Los Angeles

1. Introduction

Fix a complex Banach space $X$ with open unit ball $B$. We are interested in studying the uniform algebra $H^*(B)$ of bounded analytic functions on $B$, and its spectrum $\mathcal{A} = \mathcal{A}(B)$ consisting of the nonzero complex-valued homomorphisms of $H^*(B)$. The restriction of any $\varphi \in \mathcal{A}$ to the dual space $X^*$ of $X$ yields a linear functional $\overline{\varphi}(\varphi)$ on $X^*$, and the projection $\varphi \rightarrow \overline{\varphi}(\varphi)$ maps $\mathcal{A}$ onto the closed unit ball $B^{**}$ of the bidual $X^{**}$ of $X$. If $X$ is one-dimensional, $B$ is the open unit disk in the complex plane, and $\pi$ is the fibering discussed by K. Hoffman in [Ho]. In this case the projection $\pi$ is one-to-one over $B$, and $\pi^{-1}$ maps $B$ homeomorphically onto an open subset of $\mathcal{A}$. This probably holds whenever $X$ is finite dimensional, and it can be proved at least whenever $B$ is a finite dimensional ball or polydisk. However, when $X$ is infinite dimensional the picture changes completely. It turns out in this case (Theorem 11.1) that the fibers $\pi^{-1}(z)$ over points of $B^{**}$ are all quite large.

To obtain information about $H^*(B)$ and $\mathcal{A}$ it is useful first to study the algebra $H_0 = H_0(X)$ of complex-valued entire functions on $X$ which are bounded on bounded sets, with the topology of uniform convergence on bounded sets. The spectrum of $H_0$ is denoted by $\mathcal{M}_0 = \mathcal{M}_0(X)$; it consists of the nonzero continuous complex-valued homomorphisms of $H_0$, with the weak topology determined by the functions in $H_0$. It turns out that $\mathcal{M}_0$ has lots of analytic structure, and in fact $\mathcal{M}_0$ is a union of complex lines. There is a natural radius function $R$ on $M_0$ so that the subset $\{ R \leq 1 \}$ of $M_0$ can be identified with the spectrum of the algebra $H_0^*(B)$ of bounded analytic functions on $B$ which are uniformly continuous.

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(**) Supported by NSF grant # DMS85-03780. Part of this work was done while the author was supported by an Alexander von Humboldt Fellowship.
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Definition
Given $U$ an open subset of a Banach space $X$, then $H_b(U)$ will be the set all all Fréchet differentiable functions such that are bounded on the bounded subsets of $U$ that have a positive distance to the boundary of $U$.

$H_b(U)$ is a Fréchet algebra when endowed with the topology of uniform convergence on the sequence

$$U_r = \{ x \in X : \|x\| \leq r \text{ and } \text{dist} (x, X \setminus U) > \frac{1}{r} \}.$$
If we denote by $A$ either $A_u(B_X)$ or $H_\infty(B_X)$ or $H_b(U)$, since $X^* \subset A$, we can define

$$\pi : A \rightarrow X^{**}$$
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by

$$\pi(\phi)(x^*) := \phi(x^*).$$
If we denote by $A$ either $A_u(B_X)$ or $H_\infty(B_X)$ or $H_b(U)$, since $X^* \subset A$, we can define

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For each $z \in X^{**}$, We will call the set $M_z(A)$ the fiber of $M(A)$ at $z$, where
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For each $z \in X^{**}$, We will call the set $\mathcal{M}_z(A)$ the fiber of $\mathcal{M}(A)$ at $z$, where

$$
\mathcal{M}_z(A) = \{ \phi \in \mathcal{M}(A) : \pi(\phi) = z \}.
$$

Let $X$ be a symmetrically regular Banach space and $U$ an open subset of $X$, then $M_b(U)$ has a Riemann analytic manifold structure.
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Let $X$ be a symmetrically regular Banach space and $U$ an open subset of $X$, then $M_b(U)$ has a Riemann analytic manifold structure.

Definition

A Banach $X$ is (symmetrically) regular Banach space if every (symmetric) operator $T : X \to X^*$ is weakly-compact.

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**Definition**

A Banach $X$ is (symmetrically) regular Banach space if every (symmetric) operator $T : X \to X^*$ is weakly-compact.

$T : X \to X^*$ is *symmetric* if

$$T(x)(y) = T(y)(x),$$

for every $x$ and $y$ in $X$. 
For each \( \phi \in M_b(U) \), there is a bounded subset
\[ U_r = \{ x \in X : ||x|| \leq r \text{ and dist}(x, X \setminus U) > \frac{1}{r} \} \]
such that \( |\phi(f)| \leq ||f||_{U_r} \) for all \( f \in H_b(U) \).
For each $\phi \in M_b(U)$, there is a bounded subset $U_r = \{ x \in X : \| x \| \leq r \text{ and } \text{dist}(x, X \setminus U) > \frac{1}{r} \}$ such that $|\phi(f)| \leq \|f\|_{U_r}$ for all $f \in H_b(U)$.

Given $\phi$ in $M_b(U)$ and $w \in X^{**}$ with $\|w\| < \frac{1}{r}$,

$$\phi^w : H_b(U) \to \mathbb{C}$$

by

$$\phi^w(f) = \sum_{n=0}^{\infty} \phi(\overline{P_n(w)}).$$
For each $\phi \in M_b(U)$, there is a bounded subset $U_r = \{ x \in X : \| x \| \leq r \text{ and } \text{dist}(x, X \setminus U) > \frac{1}{r} \}$ such that $|\phi(f)| \leq \|f\|_{U_r}$ for all $f \in H_b(U)$.

Given $\phi$ in $M_b(U)$ and $w \in X^{**}$ with $\|w\| < \frac{1}{r}$,

$$\phi^w : H_b(U) \to \mathbb{C}$$

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$$\phi^w(f) = \sum_{n=0}^{\infty} \phi\left(\widetilde{P}_n(w)\right),$$

where $\sum_{n=0}^{\infty} P_n(x)(\cdot)$ is the Taylor series expansion of $f$ at $x \in U$. 

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For each $\phi \in M_b(U)$, there is a bounded subset $U_r = \{ x \in X : \|x\| \leq r \text{ and } \text{dist}(x, X \setminus U) > \frac{1}{r} \}$ such that $|\phi(f)| \leq \|f\|_r$ for all $f \in H_b(U)$.

**Given** $\phi$ in $M_b(U)$ and $w \in X^{**}$ with $\|w\| < \frac{1}{r}$,

$$\phi^w : H_b(U) \to \mathbb{C}$$

by

$$\phi^w(f) = \sum_{n=0}^{\infty} \phi \left( \tilde{P}_n(w) \right),$$

where $\sum_{n=0}^{\infty} P_n(x)(\cdot)$ is the Taylor series expansion of $f$ at $x \in U$.

Defined $V_{\phi,\epsilon} = \{ \phi^w : \|w\| < \epsilon \}$, then the family $\mathcal{V} := \{ V_{\phi,\epsilon} : \phi \in M_b(U) \text{ and } \epsilon > 0 \}$ is a basic neighborhood system for a Hausdorff topology on $M_b(U)$. 

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Given $\phi$ in $M_b(U)$ and $w \in X^{**}$ with $\|w\| < \frac{1}{r}$,

$$\phi^w : H_b(U) \to \mathbb{C}$$

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$$\phi^w(f) = \sum_{n=0}^{\infty} \phi(\tilde{P}_n(w)),$$

where $\sum_{n=0}^{\infty} P_n(x)(\cdot)$ is the Taylor series expansion of $f$ at $x \in U$. 

Defined $V_{\phi, \epsilon} = \{\phi^w : \|w\| < \epsilon\}$, then the family $\mathcal{V} := \{V_{\phi, \epsilon} : \phi \in M_b(U) \text{ and } \epsilon > 0\}$ is a basic neighborhood system for a Hausdorff topology on $M_b(U)$. 

Moreover,

$$\pi : M_b(U) \to X^{**}$$

is a local homeomorphism over $X^{**}$ and $M_b(U)$ has a Riemann analytic structure over $X^{**}$. 
If we denote by $\mathbb{D}$ the open unit disk of $\mathbb{C}$

$$M_x(H_\infty(\mathbb{D})) = \{\delta_x\},$$

for all $x \in \mathbb{D}$.
If we denote by $\mathbb{D}$ the open unit disk of $\mathbb{C}$

$$\mathcal{M}_x(H\infty(\mathbb{D})) = \{\delta_x\},$$

for all $x \in \mathbb{D}$

But for all $x \in \mathbb{T}$

$$\beta \mathbb{N} \setminus \mathbb{N} \subset \mathcal{M}_x(H\infty(\mathbb{D})), $$
If we denote by $\mathbb{D}$ the open unit disk of $\mathbb{C}$

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But for all $x \in \mathbb{T}$

$$\beta \mathbb{N} \setminus \mathbb{N} \subset M_x(H_\infty(\mathbb{D})), $$


Suppose that $X$ is an infinite dimensional Banach space. Then the fiber $M_z(H_\infty(B))$ contains a copy of $\beta(\mathbb{N}) \setminus \mathbb{N}$, for every $z \in B^{**}$. 

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If we denote by $\mathbb{D}$ the open unit disk of $\mathbb{C}$

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Suppose that $X$ is an infinite dimensional Banach space. Then the fiber $\mathcal{M}_z(H_\infty(B))$ contains a copy of $\beta(\mathbb{N}) \setminus \mathbb{N}$, for every $z \in \overline{B^{**}}$.

Aron, Cole and Gamelin 1991

$$\mathcal{M}(A_u(B_{c_0})) = \{\tilde{\delta}_z : z \in \overline{B_{\ell_\infty}}\}.$$
If we denote by $\mathbb{D}$ the open unit disk of $\mathbb{C}$

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Suppose that $X$ is an infinite dimensional Banach space. Then the fiber $M_z(H_\infty(B))$ contains a copy of $\beta(\mathbb{N}) \setminus \mathbb{N}$, for every $z \in \overline{B}^{**}$.

Aron, Cole and Gamelin 1991

$$M(A_u(B_{c_0})) = \{\tilde{\delta}_z : z \in \overline{B}_{\ell_\infty}\}.$$


Actually $\beta\mathbb{N} \setminus \mathbb{N} \subset M_x(A_u(B_{\ell_p})), $ for any $1 < p < \infty$, for every $x \in B_{\ell_p}$.

For every \( f \in H^\infty(B_{c_0}) \) and \( z \in \overline{B}_{\ell_\infty} \),

\[
\{ \varphi(f) : \varphi \in M_z(H^\infty(B_{c_0})) \cap \overline{\{ \delta_x : x \in B_{c_0} \}}^{w^*} \} = \{ \varphi(f) : \varphi \in M_z(H^\infty(B_{c_0})) \}.
\]
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For every $f \in H^\infty(B_{c_0})$ and $z \in \overline{B}_{\ell_\infty}$,

$$\{ \varphi(f) : \varphi \in \mathcal{M}_z(H^\infty(B_{c_0})) \cap \{ \delta_x : x \in B_{c_0} \}^{w^*} \} = \{ \varphi(f) : \varphi \in \mathcal{M}_z(H^\infty(B_{c_0})) \}.$$ 

Theorem. The weak Theorem holds for $A(B_{\ell_2})$.

For $f \in A_u(B_{\ell_2})$, we have

$$\hat{f}(\mathcal{M}_z(A_u(B_{\ell_2})) \cap \{ \delta_x : x \in B_{\ell_2} \}^{w^*}) = \hat{f}(\mathcal{M}_z(A_u(B_{\ell_2}))),$$

for every $z \in \hat{B}_{\ell_2}$. 
Theorem, R. Aron, D. Carando, S. Lassalle M.M. 2016
If \( x_0 \in \ell_1 \) and \( \|x_0\| = 1 \) then \( \mathcal{M}_{x_0}(A_u(B_{\ell_1})) = \{\delta_{x_0}\} \).
Theorem, R. Aron, D. Carando, S. Lassalle M.M. 2016

- If \( x_0 \in \ell_1 \) and \( \|x_0\| = 1 \) then \( \mathcal{M}_{x_0}(A_u(B_{\ell_1})) = \{\delta_{x_0}\} \).
- For all \( x_0 \in B_{\ell_1} \), then \( \beta^N \) is embedded in \( \mathcal{M}_{x_0}(A_u(B_{\ell_1})) \).
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Theorem, R. Aron, D. Carando, S. Lassalle M.M. 2016

- If $x_0 \in \ell_1$ and $\|x_0\| = 1$ then $\mathcal{M}_{x_0}(A_u(B_{\ell_1})) = \{\delta_{x_0}\}$.
- For all $x_0 \in B_{\ell_1}$, then $\beta\mathbb{N}$ is embedded in $\mathcal{M}_{x_0}(A_u(B_{\ell_1}))$.
- For every $z \in B_{\ell_1^{**}}$ we have $\text{Card}(\mathcal{M}_z(A_u(B_{\ell_1}))) \geq c$. 


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Theorem, R. Aron, D. Carando, S. Lassalle M.M. 2016

- If $x_0 \in \ell_1$ and $\|x_0\| = 1$ then $M_{x_0}(A_u(B_{\ell_1})) = \{\delta_{x_0}\}$.
- For all $x_0 \in B_{\ell_1}$, then $\beta^N$ is embedded in $M_{x_0}(A_u(B_{\ell_1}))$.
- For every $z \in B_{\ell_1^{**}}$ we have $\text{Card}(M_z(A_u(B_{\ell_1}))) \geq c$.
- There exist $z \in S_{\ell_1^{**}}$ such that $\text{Card}(M_z(A_u(B_{\ell_1}))) \geq c$,

where $c$ stands for the cardinal of the continuum.
Example R. Aron, D. Carando, T. Gamelin, S. Lassalle M.M. 2012

Let \( \{a_j \mid j \in \mathbb{N}\} \) be a dense sequence in \( \mathbb{D} \), and let \( f : B_{\ell_2} \to \mathbb{C} \) be given by \( f(x) = \sum_{n=1}^{\infty} a_n x_n^2 \). Let \( (n_j) \) be an arbitrary subsequence of natural numbers, and let \( \varphi \in \{\delta_{e_{n_j}} \mid j \in \mathbb{N}\} \) be an accumulation point in \( \mathcal{M}(A_u(B_{\ell_2})) \). Then \( \pi(\varphi) = 0 \) for every such \( \varphi \). In addition, for any \( \lambda_0 \in \mathbb{D} \), if \( (n_j) \) is chosen so that \( a_{n_j} \to \lambda_0 \), then the corresponding \( \varphi \) satisfies \( \varphi(f) = \lambda_0 \). Hence

\[
\mathbb{D} \subset \hat{f}((\mathcal{M}_0((A_u(B_{\ell_2})))).
\]
Injecting analytic structures
Maximal Ideals in an Algebra of Bounded Analytic Functions

I. J. SCHARK

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1. Introduction. Let $D$ be the open unit disc in the complex plane, and let $B$ be the algebra of bounded analytic functions on $D$. Using the uniform norm, $B$ is a commutative Banach algebra which has attracted considerable attention in recent years. In this paper we shall present various results concerning the maximal ideal space of $B$. The results were obtained during the Conference on Analytic Functions held at the Institute for Advanced Study in 1957.

The structure of the paper is as follows. Section 2 introduces the space $\mathcal{H}$ of complex homomorphisms (maximal ideals) of the algebra $B$, as well as the Gelfand isomorphism $\mathcal{H} \rightarrow B$ with a uniformly closed algebra of continuous functions on $\mathcal{H}$. There is a natural projection $\pi$ of $\mathcal{H}$ onto the closed disc in the plane, obtained by sending each complex homomorphism into its value on the coordinate function $z$. This map $\pi$ is one-one over the open disc $D$, and shows that the natural injection of $D$ into $\mathcal{H}$, which sends $\lambda$ into the homomorphism “evaluation at $\lambda^n$,” is a homeomorphism of $D$ onto an open subset $\Delta$ of $\mathcal{H}$. The remaining closed set of homomorphisms is mapped by $\pi$ onto the unit circle $C$. This closed set $\mathcal{H} \setminus \Delta$ is decomposed by $\pi$ into disjoint closed fibers $\mathcal{H}_n$, where for $|\alpha| = 1$

$$\mathcal{H}_n = \{\varphi \in \mathcal{H}; \varphi(\alpha) = \alpha\} = \pi^{-1}(\alpha).$$

Through the action of the rotation group of the plane on $B$, one sees that the fibers $\mathcal{H}_n$ are homeomorphic with one another.

In Section 3 we identify the Silov boundary for the algebra $B$. Its description is as follows. A theorem of Farlow enables one to identify $B$ with a closed subalgebra of the algebra $L^\infty$ of essentially bounded measurable functions on the unit circle. This gives a natural continuous map $\tau$ of the (extremely disconnected) space of maximal ideals of $L^\infty$ into the space $\mathcal{H}$. We show that $\tau$ is a homeomorphism, the range of which is the Silov boundary for $B$. It is observed that the Silov boundary is a subset of $\Delta \setminus \Delta$, but does not exhaust $\mathcal{H} \setminus \Delta$.

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1 A pseudonym for a large group of mathematicians who discussed these problems during the 1957 Conference on Analytic Functions sponsored by the Institute for Advanced Study under contract No. AF 18(603)-118 with the Air Force Office of Scientific Research.

The complex disk $\mathbb{D}$ can be analytically and homeomorphically injected into $\mathcal{M}_1(H^\infty(\mathbb{D}))$. 

Theorem I. J. Schark 1961
In summary, we have constructed a homeomorphism $\varphi$ of the open disc $D$ into the fiber $\mathfrak{C}_1$, and $\varphi$ is analytic, in the sense that $\hat{f} \circ \varphi$ is analytic for every $f \in B$. Thus the disc $\varphi(D)$ has a natural analytic structure, and when we restrict the algebra $\hat{B}$ to this disc, we obtain the algebra of all bounded analytic functions on $\varphi(D)$. It is easy to see that the uniformly closed restriction algebra $\hat{B}|_{\varphi(D)}$ will have as its maximal ideal space the subset $S$ of $\mathfrak{C}$ defined by

$$S = \{ \varphi \in \mathfrak{C}; |\varphi(f)| \leq \sup_{\varphi(D)} |\hat{f}|, \text{ all } f \in B \}.$$ 

This set $S$ is contained in $\mathfrak{C}_1$, as we see by considering $f(\lambda) = \frac{1}{2}(1 + \lambda)$. Since the restriction algebra is isomorphic to the algebra of bounded analytic functions in the disc, the set $S$ must be homeomorphic to the entire maximal ideal space $\mathfrak{C}$.

The maximum modulus principle makes it clear that $\varphi(D)$ lies in $\mathfrak{C}_1$, and so we see more vividly than before that $\overline{A} - \Delta \neq \Gamma$. One sees from the above discussion that the space $\mathfrak{C}$ “reproduces” itself in any given fiber ad infinitum. Because in $S$ there are fibers attached to the disc $\varphi(D)$ in each of which is a closed set homeomorphic to $\mathfrak{C}$, and so on.
Theorem: Cole, Gamelin and Johnson 1992

If $X = \ell_2$, then the unit ball of a nonseparable Hilbert space injects into the fiber $\mathcal{M}_0(H^\infty(B))$ via an analytic map which is uniformly bicontinuous from the metric of the unit ball of the Hilbert space to the Gleason metric of its image in $\mathcal{M}(H^\infty(B))$. 

Gleason metric in $\mathcal{M}(H^\infty(B))$ is defined as

$$\rho(\phi,\psi) = \sup\{ |\hat{f}(\phi) - \hat{f}(\psi)| : f \in H^\infty(B), \|f\| \leq 1 \}$$
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Introduction. The algebras of holomorphic functions in $\mathbb{C}^N$.
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Theorem: Cole, Gamelin and Johnson 1992

Suppose that $X$ has a normalized basis $(e_j)$ that is shrinking, with associated functionals $(e_j^*)$ satisfying that there exists a positive integer $N \geq 1$ such that

$$ \sum_{j=1}^\infty |e_j^*(x)|^N < \infty $$

for all $x = \sum_{j=1}^\infty e_j^*(x)x_j$ in $X$. Then there is an analytic injection of the countable infinite dimensional polydisk $\mathbb{D}^N$ into the fiber $\mathcal{M}_0(H^\infty(B))$. 

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Theorem: Cole, Gamelin and Johnson, 1992

Let $X$ be an infinite dimensional Banach space. Suppose that $(z_k)$ is a sequence in $B^{**}$ which converges weak-star to 0, such that the distance from $z_k$ to the linear span of $z_1, \ldots, z_{k-1}$ tends to 1 as $k \to \infty$ (e.g. $c_0$ and $\ell_p$ for $1 < p < \infty$). Then, passing to a subsequence, we can find a sequence of analytic disks $\lambda \to z_k(\lambda)$ ($\lambda \in \mathbb{D}, \ k \geq 1$) in $B^{**}$ with $z_k(0) = z_k$, such that for each $\lambda \in \mathbb{D}$, $(z_k(\lambda))$ is an interpolating sequence for $H^\infty(B)$. Furthermore, the correspondence $(k, \lambda) \to z_k(\lambda)$ extends to an embedding

$$
\Psi : \beta(\mathbb{N}) \times \mathbb{D} \to \mathcal{M}(H^\infty(B))
$$

such that

$$
\Psi((\beta(\mathbb{N}) \setminus \mathbb{N}) \times \mathbb{D}) \subset \mathcal{M}_0(H^\infty(B))
$$

and $\hat{f} \circ \Psi$ is analytic on each slice $\{p\} \times \mathbb{D}$ for all $f \in H^\infty(B)$ and $p \in \beta(\mathbb{N})$. 
Remark

Let $U$ be an open subset of a Banach space and $X$ and $\phi \in M_b(U)$. There exists $r > 0$ such that

$$F : B_{X^{**}}(0, \frac{1}{r}) \to M_b(U),$$

$$F(w) = \phi^w : H_b(U) \to \mathbb{C},$$

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$$|\phi(f)| \leq \sup_{x \in U_r} |f(x)|,$$

for all $f \in H_b(U)$, where $U_r = \{x \in X : \|x\| \leq r \text{ and dist } (x, X \backslash U) > \frac{1}{r}\}$. 
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Recall

$$\phi^w(f) = \sum_{n=0}^{\infty} \phi\left(\bar{P}_n(w)\right),$$

where $\sum_{n=0}^{\infty} P_n(x)$ is the Taylor series expansion of $f$ about $x \in U$. 
Aron, Cole and Gamelin 1991

\[ \mathcal{M}(H_u(B_{c_0})) = \{ \tilde{\delta}_z : z \in \ell_\infty \}. \]
Introduction. The algebras of holomorphic functions in $\mathbb{C}^N$. Infinite dimensional setting. Size of the fibres. Injecting analytic structures.

**Proposition, R. Aron, J. Falcó, D. García and M.M, 2016**

Let $X \neq \{0\}$ be a Banach space and $x_0 \in S_X$. Then the complex disk $\mathbb{D}$ can be analytically injected into $\mathcal{M}_{x_0}(H^\infty(B_X))$. 

**Theorem, R. Aron, J. Falcó, D. García and M.M, 2016**

Let $z_0 = (z_1, z_2, ..., z_n, ...)$ be a point of the distinguished boundary $\mathbb{T}_{\aleph_0}$ of $B_{\ell^\infty}(i.e |z_j| = 1$ for all $j \in \mathbb{N})$. Then there exists an injection $\Psi: B_{\ell^\infty} \rightarrow \mathcal{M}_{z_0}(H^\infty(B_{c_0}))$ which is biholomorphic onto its image.

Let $K$ be an infinite scattered compact Hausdorff set. We have that $B_{\ell^\infty}$ is continuously injected in the fiber $\mathcal{M}_{0}(H^\infty(B_K))$. 

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Analytic structures in maximal ideal spaces

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Let $K$ be an infinite scattered compact Hausdorff set. We have that $B_{\ell_\infty}$ is continuously injected in the fiber $\mathcal{M}_0(H^\infty(B_{c(K)})).$

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Let $X$ be a Banach space such that there exists a polynomial $P$ satisfying that $P|_{B_X}$ is not weakly continuous at some point of $B_X$. Then the complex disk $\mathbb{D}$ can be analytically injected in $\mathcal{M}_z(A_u(B_X))$ for every $z \in B_X^{**}$.

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Remark

If every polynomial is weakly continuous at each point of $B_X$ and $X^*$ has the approximation property, then

$$M_z(A_u(B_X)) = \{\tilde{\delta}_z\}$$

for every $z \in \overline{B_{X^{**}}}$.


