The search for linearity in linear dynamics

Karl Grosse-Erdmann

Département de Mathématique
Université de Mons, Belgium

Infinite Dimensional Analysis
Celebrating Richard Aron’s Work and Impact
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Linear dynamics

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Linear dynamics studies the behaviour of orbits of (continuous, linear) operators.

Throughout the talk, let

\[ T : X \rightarrow X \]

be an operator on a separable Banach space (or Fréchet space) \( X \).
The operator $T$ is **hypercyclic** if there is a vector $x \in X$ such that

$$\text{orb}(x, T) = \{x, Tx, T^2x, \ldots\}$$

is dense in $X$. 

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The search for linearity

Celebrating Richard Aron
The operator $T$ is hypercyclic if there is a vector $x \in X$ such that

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is dense in $X$.

Each such vector $x$ is then called a hypercyclic vector for $T$. We denote by

$$HC(T)$$

the set of all hypercyclic vectors.
Three classical examples of hypercyclic operators:

- Birkhoff (1929)
  \( X = H(C) \): space of entire functions, compact-open topology
  \( T_a : f \rightarrow f(\cdot + a) \), \( a \neq 0 \): the translation operators

- MacLane (1952)
  \( X = H(C) \)
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  \( X = \ell^p, 1 \leq p < \infty \), or \( c_0 \)
  \( \lambda B : (x_n) \rightarrow \lambda (x_n + 1) \), \( |\lambda| > 1 \): multiples of the backward shift
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The set $HC(T)$ of hypercyclic vectors is fundamentally non-linear.

It is a well-known consequence of the Baire category theorem that the set $HC(T)$ is always residual if $T$ is hypercyclic, which then implies that

$$X = HC(T) + HC(T).$$

And, of course, for most operators many non-zero vectors are not hypercyclic.
Put in a different way, there is no good reason why the sum of two hypercyclic vectors should be hypercyclic.
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But does $HC(T)$ contain a dense linear subspace (except for 0)?

Beauzamy (Studia Math. 1987, 8 pp.): There is a hypercyclic operator $T$ on Hilbert space $X$ and a hypercyclic vector $x \in X$ such that

$$\mathbb{Q} - \text{span}\{x, Tx, T^2x, \ldots\} \setminus \{0\} \subset HC(T).$$
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Beauzamy (Studia Math. 1990, 9 pp.): Can even do it for the full span.

So, for some $T$, the set of hypercyclic vectors is densely lineable.
But we have:

**Theorem (Herrero-Bourdon-Bès 1991-1999)**

*For any hypercyclic operator $T$, $HC(T)$ is densely lineable.*
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*For any hypercyclic operator $T$, $HC(T)$ is densely lineable.*

In fact, for any hypercyclic vector $x \in X$ for any operator $T$, we have that

$$\text{span}\{x, Tx, T^2x, \ldots\} \setminus \{0\} \subset HC(T)$$

(essentially by a Hahn-Banach argument).
Common lineability

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So can two operators share a dense linear subspace (apart from 0) of hypercyclic vectors?

Problem (Aron 2001)

Do the operators $D$ and $T_1$ admit a dense linear subspace of $H(\mathbb{C})$ so that each non-zero vector in the subspace is hypercyclic for $D$ and for $T_1$?
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*Do the operators $D$ and $T_1$ admit a dense linear subspace of $H(\mathbb{C})$ so that each non-zero vector in the subspace is hypercyclic for $D$ and for $T_1$?*

Grivaux observed in 2003 that the answer is yes as a consequence of the proof of the Herrero-Bourdon-Bès theorem since the two operators commute.
But she proved that the result remains true even if the operators do not commute:

**Theorem (Grivaux 2003)**

Let $T_\nu$, $\nu \geq 1$, be hypercyclic operators on a Banach space $X$. Then there exists a dense linear subspace of $X$ all of whose non-zero vectors are hypercyclic for any $T_\nu$, $\nu \geq 1$.

The proof uses the Baire category theorem on the space $B(X)$ of bounded linear operators on $X$. 
Can one be more ambitious:

**Definition**

A hypercyclic subspace for an operator $T$ on $X$ is an infinite-dimensional closed subspace of $X$ such that each of its non-zero vectors is hypercyclic for $T$. In other words, $\text{HC}(T)$ is spaceable if and only if $T$ admits a hypercyclic subspace.
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So, do hypercyclic subspaces exist?

The Rolewicz operators $T = \lambda B$, $|\lambda| > 1$, on $\ell^p$, $p \geq 1$, or $c_0$, do not admit hypercyclic subspaces.
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Some operators do not support hypercyclic subspaces.

Theorem (Montes 1996)

The Rolewicz operators $T = \lambda B$, $|\lambda| > 1$, on $\ell^p$, $p \geq 1$, or $c_0$, do not admit hypercyclic subspaces.
However, there are operators that do possess hypercyclic subspaces. For complex Banach spaces, after considerable work, the following characterization was found.

**Theorem (González-León-Montes 2000)**

Let $T$ be an operator on a complex Banach space $X$ such that $T \oplus T$ is hypercyclic. Then the following assertions are equivalent:

1. $T$ has a hypercyclic subspace;
2. there exists an increasing sequence $(n_k)_{k}$ of positive integers and an infinite-dimensional closed subspace $M_0$ of $X$ such that $T^{n_k}x \to 0$ for all $x \in M_0$.

The proof uses spectral theory in an essential way. We are still missing a characterization for real Banach spaces or for Fréchet spaces.
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The proof uses spectral theory in an essential way. We are still missing a characterization for real Banach spaces or for Fréchet spaces.
How about Fréchet space operators?

**Theorem (Bernal-Montes 1995)**

*The translation operators $T_a, f \mapsto f(\cdot + a), a \neq 0$, on $H(\mathbb{C})$ admit hypercyclic subspaces.*
So, among the classical hypercyclic operators we were still left with the differentiation operator $D$.

**Problem (Aron, published in 2007)**

*Does the differentiation operator $D : f \rightarrow f'$, on $H(\mathbb{C})$ admit a hypercyclic subspace?*

Richard: *"In a rough sense, the operator $D$ behaves like the Rolewicz operator, and so a first guess might be that $HC(D)$ does not contain a closed infinite-dimensional subspace. But guesses are rarely publishable!"

The answer was found by Shkarin.

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More generally, if

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an entire function of exponential type, that is, there are $M, A > 0$ such that $|\varphi(z)| \leq Me^{A|z|}$ for all $z \in \mathbb{C}$, then

$$\varphi(D)f = \sum_{n=0}^{\infty} a_n D^n f$$

defines an operator $\varphi(D)$ on $H(\mathbb{C})$. 

Godefroy and Shapiro had shown in 1991 that, for any non-constant entire function $\varphi$ of exponential type, the operator $\varphi(D)$ is hypercyclic.
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Thus the following generalizes Bernal-Montes.

**Theorem (Petersson 2006)**

If $\varphi$ is an entire function of exponential type that is not a polynomial, then $\varphi(D)$ admits a hypercyclic subspace.
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The following generalizes Shkarin and thus completes the picture.

**Theorem (Menet 2014)**

*For any non-constant polynomial \( P \), the operator \( P(D) \) admits a hypercyclic subspace.*
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**Theorem (Menet 2014)**

For any non-constant polynomial $P$, the operator $P(D)$ admits a hypercyclic subspace.

In fact, Menet has obtained powerful necessary conditions and sufficient conditions for Fréchet space operators to have hypercyclic subspaces.
Common spaceability

Aron et al. present a counterexample \( \ell_2 \oplus \ell_2 \) of the form

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T_1 = (I + Bw) \oplus 2B \\
T_2 = 2B \oplus (I + Bw)
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where \( Bw \) is a certain weighted backward shift.
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R. Aron, J. Bès, F. León and A. Peris 2005 address the natural question of common spaceability:

If two operators on the same space admit hypercyclic subspaces do they have a common hypercyclic subspace?

Aron et al. present a counterexample $\ell^2 \oplus \ell^2$ of the form $T_1 = (I + Bw) \oplus 2B$ and $T_2 = 2B \oplus (I + Bw)$, where $Bw$ is a certain weighted backward shift.
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**Theorem (Aron-Bès-León-Peris 2005)**

Let $T_\nu, \nu \geq 1$, be operators on a Banach $X$. If there exists an increasing sequence $(n_k)_k$ of positive integers such that

- each $T_\nu, \nu \geq 1$, satisfies the Hypercyclicity Criterion for $(n_k)$,
- there is an infinite-dimensional closed subspace $M_0$ of $X$ such that $T_\nu^{n_k}x \to 0$ for all $x \in M_0$ and all $\nu \geq 1$, 

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The result remains true for Fréchet spaces whose topology is induced by a sequence of norms.

**Example**

Any two translation operators \( T_a, T_b, a, b \neq 0 \), on \( H(\mathbb{C}) \) have a common hypercyclic subspace.
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**Theorem (Aron-Bès-León-Peris 2005)**

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**Example**

Any two translation operators $T_a$, $T_b$, $a, b \neq 0$, on $H(\mathbb{C})$ have a common hypercyclic subspace.

It seems to be not known if $D$ and $T_a$ have a common hypercyclic subspace on $H(\mathbb{C})$...
Algebrability

Having found large subspaces of hypercyclic vectors for many operators, it is natural to look for algebras of such vectors – provided the Banach or Fréchet space has an algebraic structure.

Definition

A hypercyclic algebra for an operator $T$ on $X$ is a subalgebra of $X$ so that each of its non-zero vectors is hypercyclic for $T$.

The research in this area was initiated by R. Aron.

An easier question is whether there is a hypercyclic vector $x$ so that each power $x^k$, $k \geq 1$, remains hypercyclic.
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The following is a striking application of Hurwitz’s theorem.

**Theorem (Aron-Conejero-Peris-Seoane 2007)**

Let $T_a$, $a \neq 0$, be a translation operator on $H(\mathbb{C})$, let $f \in H(\mathbb{C})$, $k \geq 1$. If 

$$g \in \text{orb}(f^k, T_a)$$

then the order of each zero of $g$ is a multiple of $k$. 

*In particular, no square $f^2$ can be hypercyclic.*
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The authors have a partial positive result for the differentiation operator $D$ on $H(\mathbb{C})$.

**Theorem (Aron-Conejero-Peris-Seoane 2007)**

There is a function $f \in H(\mathbb{C})$ such that each power $f^k$, $k \geq 1$, is hypercyclic for $D$. The set of such functions is residual in $H(\mathbb{C})$. 
But the full question is that for the existence of a hypercyclic algebra:

**Problem (Aron 2007)**

*Does D admit a hypercyclic algebra? In other words, is there an entire function f so that any function of the form*

\[ \sum_{k=0}^{n} a_k f^k \neq \text{const.} \]

*is hypercyclic for D?*
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**Problem (Aron 2007)**

Does $D$ admit a hypercyclic algebra? In other words, is there an entire function $f$ so that any function of the form

$$\sum_{k=0}^{n} a_k f^k \neq \text{const.}$$

is hypercyclic for $D$?

The positive answer was given independently in two publications.

**Theorem (Bayart-Matheron 2009, Shkarin 2010)**

Yes!
We know from Godefroy and Shapiro that for any non-constant entire function $\varphi$ of exponential type, $\varphi(D)$ is also hypercyclic. Now, $T_a = \exp(aD)$ does not have a hypercyclic algebra, so it is natural to restrict attention to polynomials.

**Problem (Aron)**

Does $P(D)$, $P$ a non-constant polynomial, admit a hypercyclic algebra?
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**Problem (Aron)**

Does \( P(D) \), \( P \) a non-constant polynomial, admit a hypercyclic algebra?

This question has recently been studied.

**Theorem (Bès-Conejero-Papathanasiou, JMAA 15 Jan. 2017)**

Yes, if \( P(0) = 0 \)!
One may study lineability in Linear Dynamics in yet another way.

If $T$ is a hypercyclic operator on a Banach space then necessarily $\|T\| > 1$. So no operator can have all its (non-zero) multiples hypercyclic.

However, for the differentiation operator $D$ on the Fréchet space $H(C)$, every operator $\lambda D$, $\lambda \neq 0$, is hypercyclic.

Problem (Aron, no place no date) If $X$ is a separable non-normable Fréchet space, is the set of hypercyclic operators lineable?

The specific question if every such Fréchet space admits an operator $T$ such that $\lambda T$ is hypercyclic for any $\lambda \neq 0$ was published by Bonet (2010).
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Richard’s publications in Linear Dynamics


A personal remark

In the last 20 years, when I met Richard at conferences, I was often reminded of the German fairy tale

*Der Hase und der Igel*
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Karl Grosse-Erdmann (UMons)

The search for linearity

Celebrating Richard Aron 25 / 27
A personal remark

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*Der Hase und der Igel*

In a race between the hedgehog and the hare, the hedgehog wins by employing a trick.

He positions his wife at the endpoint of the race, and when the hare arrives there she comes forward, saying

*Bin schon hier* (I am here already)
In the past I very often felt like the hare when I went to a conference, be it in Spain, in the US, or at other places: Richard was already there!

Of course, the analogy breaks down there. First, Richard didn’t need to ask Eleanor to stand in for him. And, most importantly, I was always delighted to meet him.

I am looking forward to many more of the Hase-und-Igel moments in the future.
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