Functions of Real Variables 1 (62051/72051)
Home Work 8, due on Monday October 31.
Instructor: Prof. Artem Zvavitch.

I WILL ADD MORE PROBLEMS ON WEDNESDAY

Problem 1. Prove that if $f \in L^1(\mathbb{R}^d)$ is not identically zero, then

$$f^*(x) \geq \frac{c}{|x|^d},$$

for some $c > 0$ and all $|x| \geq 1$. Conclude that $f^*$ is not integrable on $\mathbb{R}^d$. Then, show that the weak type estimate

$$m(\{x : f^*(x) > \alpha\}) \leq \frac{c}{\alpha}$$

for all $\alpha > 0$, whenever $\int |f| = 1$ is best possible in the following sense: if $f$ is supported in the unit ball with $\int |f| = 1$, then

$$m(\{x : f^*(x) > \alpha\}) \geq \frac{c'}{\alpha}$$

for some $c' > 0$ and all sufficiently small $\alpha$.

I would suggest to start with an observation that if $f$ is not identically zero then there always exists a ball $B$ such that $\int_B |f| > 0$.

Problem 2. Consider the function on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{|x| (\log 1/|x|)^2}, & \text{if } |x| \leq 1/2, \\ 0, & \text{otherwise} \end{cases}$$

• Show that $f$ is integrable.

• Establish the inequality

$$f^*(x) \geq \frac{c}{|x| (\log 1/|x|)}$$

for some $c > 0$ and $|x| \leq 1/2$, to conclude that the maximal function $f^*$ is not locally integrable.