Introduction to Partial Differential Equations.
Home Work 1, due Wednesday, February 4.
Instructor: Prof. Artem Zvavitch

Problem 1. Consider polar coordinates coordinates \((r, \theta)\), where \(r\) is the distance from the origin, \(\theta\) is the angle in \(xy\)-plane from \(x\)-axis.

- Show that the standard coordinates can be rewritten as 
  \[ x(r, \theta) = r \cos \theta, \]
  \[ y(r, \theta) = r \sin \theta, \]
- Notice that \(r^2 = x^2 + y^2\) apply to this equality \(\frac{\partial}{\partial x}\) to find \(\frac{\partial r}{\partial x}\).
  Also find \(\frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}\) and \(\frac{\partial \theta}{\partial y}\).
- Consider function \(u(x, y)\). Use the chain rule to prove that 
  \[ \frac{\partial u}{\partial x} = \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta}, \]
  \[ \frac{\partial u}{\partial y} = \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta}. \]
- Now you are ready to compute the Laplacian in polar coordinates, prove that 
  \[ \nabla^2 u = \Delta u = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}. \]

Problem 2. Show that the following operators are linear:
- \(Lu = \Delta u\).
- \(Lu = \Delta \left[ K_0(x, y, z) u(x, y, z) \right].\)
- \(Lu = \int_0^2 u(y) dy.\)

Problem 3. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?
- \(\Delta u(x, y) = 0.\)
- \(\frac{\partial u}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right)\)

Problem 4. Solve heat equation \(\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}\) subject to \(u(0, t) = 0, u(L, t) = 0\) and \(u(x, 0) = 3 \sin \frac{\pi x}{L} - 2 \sin \frac{4 \pi x}{L}.\)