# Topics in entire functions. Lectures in Kent University

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*Preliminary version*

**1. Real polynomials with real zeros, Laguerre–Pólya class and $\mathfrak{R}$-functions.**

Consider the class of real polynomials $P$ with all zeros real.

**Theorem.** *This class is closed under differentiation.*

*Proof.* Let $d = \deg P$. Then $\deg P' = d-1$, so $P'$ has at most $d-1$ zeros (counting multiplicity). On the other hand, it follows from Rolle's theorem that $P'$ has at least $d-1$ real zeros (counting multiplicity!). So all zeros of $P'$ are real.

This is a *non-trivial* proof! We used the following principle: if $X \subset Y$ are finite subsets, and $\operatorname{card} X \geq \operatorname{card} Y$ then $X = Y$. And the degree $d$, which was used in the proof, is not mentioned in the statement of the theorem.

Entire functions are limits of polynomials (uniform on compact subsets in $\mathbf{C}$). Can this theorem be generalized to entire functions? Not in a straightforward way:

$$f(z) = ze^{z^2/2}, \quad f'(z) = (z^2+1)e^{z^2/2}.$$

**Definition.** *Laguerre–Pólya class LP consists of entire functions which are limits of real polynomials whose all zeros are real.*

**Exercise.** *Let $P_n$ be a sequence of real polynomials with all zeros real. Suppose that $P_n \to f \not\equiv 0$ uniformly in a neighborhood of some real point. Then $P$ is entire and convergence holds on all compacts in the plane.*

Class $LP$ is closed under differentiation. It is also closed with respect to multiplication and taking limits.

**Examples.**

$$\sin \pi z = z \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 - \frac{z^2}{k^2}\right), \quad e^z = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n,$$

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) e^{-z/k},$$

$$e^{-z^2} = \lim_{n\to\infty} \left(1 - \frac{z^2}{n}\right)^n,$$

however $e^{z^2} \notin LP$, as we have seen above. Other examples are $\cos \sqrt{z}$ and $(\sin \sqrt{z})/\sqrt{z}$ which are in $LP$.

**Theorem.** (Laguerre–Pólya) *$LP$ consists of entire functions of the form*

$$f(z) = c z^m e^{az^2+b} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{z/z_k}, \tag{1}$$

*where $c$, $b$ and $z_k \neq 0$ are real, $m \geq 0$ is an integer, $a \leq 0$, and*

$$\sum_k |z_k|^{-2} < \infty, \tag{2}$$

*so that the infinite product is absolutely convergent.*

This is an example of a *parametric description*: a formula which gives all functions of a class in terms of certain parameters whose domain is explicitly described.

**Definition.** *A function $\phi$ analytic in $\mathbf{C}\backslash\mathbf{R}$ is called an $\mathfrak{R}$-function if $\phi(\overline{z}) = \overline{\phi(z)}$, and $\operatorname{Im} \phi(z) \operatorname{Im} z \geq 0$, $z \in \mathbf{C}\backslash\mathbf{R}$.*

**Exercise.** *A parametric description of the class of rational $\mathfrak{R}$-functions is this:*

$$\phi(z) = az + b - \sum_k \frac{c_k}{z - z_k}, \tag{3}$$

*where $a \geq 0,\ c_k > 0,$ and $z_k, b$ are real.*

**Exercise.** *Prove that $\mathfrak{R}$ is a normal family in the upper half-plane $H$: from every sequence of functions of class $\mathfrak{R}$ one can select a subsequence that converges to a function in $R$ or to $\infty$ uniformly on compact subsets of $H$. Hint: the family of analytic functions mapping the unit disk into itself is a normal family.*

**Exercise.** *A parametric description of $\mathfrak{R}$-functions meromorphic in $\mathbf{C}$ is*

$$\phi(z) = az + b - \frac{c_0}{z} - \sum_k c_k \left( \frac{1}{z - z_k} + \frac{1}{z_k} \right) \tag{4}$$

*where $a \geq 0,\ c_k \geq 0,\ z_k, b$ are real, and*

$$\sum_k \frac{c_k}{|z_k|^2} < \infty.$$

*Hint: how positive harmonic functions in the upper half-plane look? See section 4.*

*Proof of the Laguerre–Polya theorem.* If $f$ is given by (1) then $f \in LP$ in view of the examples given above. Suppose now that $P_n$ are real polynomials with all zeros real, $P_n \to f$. Assuming wlog that $f(0) \neq 0$, we have

$$P_n(z) = c_n \prod_{k=1}^{d_n} \left( 1 - \frac{z}{z_{n,k}} \right),$$

so

$$\frac{P_n'(z)}{P_n(z)} = \sum_{k=1}^{d_n} \frac{1}{z - z_{n,k}}, \tag{5}$$

so $-P_n'/P_n \in \mathfrak{R}$. Notice that

$$\left( P_n'/P_n \right)' (0) = \sum_{k=1}^{d_n} z_{n,k}^{-2},$$

so the sums in the RHS are bounded as $n \to \infty$, thus there cannot be many zeros on any finite interval, since all summands are positive, and we have the limit sequence $(z_k)$ which enjoys the same property

$$\sum_k z_k^{-2} < C. \tag{6}$$

Thus the limit of $-P_n'/P_n$ is meromorphic in $\mathbf{C}$, so it is represented by (4) with integer $c_k$. By integrating (4) and exponentiating we obtain (1).

All $LP$ functions have order at most 2, normal type, which means

$$|f(z)| \leq Ce^{A|z|^2}, \quad z \in \mathbf{C}$$

with some $C, A > 0$ and every real entire function of order less than 2 with all zeros real is an LP function by the Hadamard factorization theorem.

We finish this section with mentioning a geometric interpretation of $LP$ functions. Recall that by Rolle's theorem zeros of a rational $R$-function are interlacent with the poles.

**Exercise**. *Show that the formula*

$$f(z) = c\frac{z-a_0}{z-b_0}\prod_{k\neq 0}\left(1-\frac{z}{a_k}\right)\left(1-\frac{z}{b_k}\right)^{-1}, \tag{7}$$

*where $b_k < a_k < b_{k+1}$, $a_{-1} < 0 < b_1$ and $c > 0$ gives a parametric representation of meromorphic functions of class $\mathfrak{R}$. (The sequences $a_k, b_k$ may be finite or infinite in one or both directions).*

In particular, for a polynomial $P \in LP$ we obtain

$$\frac{P'(z)}{P(z)} = \frac{d}{z-z_0}\prod_{k=1}^{d-1}\frac{z-w_n}{z-z_n},$$

where $z_0 \leq w_1 \leq z_1 \leq \ldots \leq w_{d-1} \leq z_{d-1}$, and $d = \deg p$. This shows that $\log P$ is the Schwarz–Christoffel map in the upper half-plane, mapping this half-plane onto region with interior angles 0 and $2\pi$. More precisely, the image of the upper half-plane is a region obtained from the horizontal strip of width $\pi d$ by removing horizontal slits $(-\infty + i\pi k, \log c_k + i\pi k]$, where $(-1)^k c_k$ are the critical values of $P$.

This is an example of a *comb domain*. In general a comb domain is obtained from a plane, or an upper or lower half-plane, or a horizontal strip by removing cuts along some rays to the left. The whole boundary of the domain is contained in the horizontal lines $y = \pi k$.

If $D$ is any comb domain (with non-empty boundary), and $\theta : H \to D$ a conformal map of the upper half-plane onto $D$, then $\exp \circ \theta$ extends to the whole plane by symmetry and becomes a Laguerre–Pólya function. This function is determined uniquely up to a real affine change of the independent variable. The comb domain depends on the free parameters $c_k$. So we obtain another parametric representation of the class $LP$, with critical values as parameters. The study of the correspondence between these different parametrizations is an interesting and non-trivial problem.

## 2. Wiman and Pólya conjectures.

What happens to the zeros of a real entire function if we differentiate it repeatedly? Pólya observed in 1944 that for functions of order $< 2$ the zeros tend to come closer to the real line, while for functions of order $> 2$ they tend to move away from the real line. He made the following two conjectures:

*Conjecture A.* If $f$ is a real entire function of order less than 2, with finitely many non-real zeros, then some derivative $f^{(n)}$ is in $LP$.

*Conjecture B.* If $f$ is a real entire function of order greater than 2, with finitely many non-real zeros, then the number of non-real zeros of $f^{(n)}$ tends to infinity with $n$.

All this has been confirmed and we are going to discuss several theorems of this kind.

Let $LP^*$ be the class of real entire functions of the form $Ph$, where $P$ is a real polynomial, and $h \in LP$. So all but finitely many zeros are real.

**Theorem 1.** (Kim) *For every $f \in LP^*$ there exists $n_0$ such that for $n \geq n_0$, $f^{(n)} \in LP$.*

This is a refined form of Conjecture 1. In the original form it was earlier proved by Czordas, Craven and Smyth. Kim extended the result to $LP^*$, with a simpler proof. His proof was further simplified by Ki and Kim.

For Conjecture 2 we have

**Theorem 2** (Bergweiler, Eremenko, Langley) *Let $f = Ph$, where $P$ is a real polynomial and $f$ a real entire function with all zeros real but $h \notin LP$. Then the number $N(f^{(n)})$ of non-real zeros of $f^{(n)}$ tends to infinity as $n \to \infty$.*

*More precisely:*

*If $f$ is of finite order then $\liminf_{n\to\infty} N(f^{(n)})/n > 0$ (Bergweiler–Eremenko), and if $f$ is of infinite order, then $N(f^{(n)}) = \infty$ for $n \geq 2$ (Langley).*

**Corollary.** *For an arbitrary real entire function $f$ there is an alternative:*

*Either* $N(f^{(n)}) = 0$ *for* $n \geq n_0$*, and* $f \in LP^*$*,*
*or* $N(f^{(n)}) \to \infty$*, and* $f \notin LP^*$*.*

This simply stated corollary is in fact a combination of three theorems with very different proofs.

One can make more precise statements about $N(f^{(n)})$ with fixed $n \geq 2$ for $f \notin LP^*$, see section 8.

The above results imply that if $f$ is a real entire function, and $f^{(n)}$ has only real zeros for all $n$, then $f \in LP$. More than 100 years ago Andres Wiman made a much more precise conjecture: if $f$ is a real entire function, and $ff''$ has only real zeros then $f \in LP$. This conjecture has been intensively studied during the last century, the finite order case was proved by T. Sheil-Small, and the infinite order case by Bergweiler, Eremenko and Langley. We discuss this in section 8. Then Langley extended the result as follows:

**Theorem 3** (Langley) *If* $f$ *is a real entire function of infinite order with finitely many non-real zeros. Then* $f^{(n)}$ *has infinitely many non-real zeros for each* $n \geq 2$*.*

This gives the infinite order case of Theorem 2.

**3. Proof of Kim's theorem.**

Let $f \in LP^*$. Then the logarithmic derivative is

$$\frac{f'(z)}{f(z)} = g(z) + \sum_{j=1}^{d} \left( \frac{1}{z - c_j} + \frac{1}{z - \overline{c_j}} \right), \tag{8}$$

where $g$ has negative real part in $H$. We use the identity

$$\text{Im} \left( \frac{1}{z - c} + \frac{1}{z - \overline{c}} \right) = \frac{-2\text{Im}\, z}{|z - c|^2 |z - \overline{c}|^2} (|z - \text{Re}\, c|^2 - (\text{Im}\, c)^2).$$

If $f'(z) = 0$ this must be positive, at least for one summand in the RHS of (8) that is $|z - \text{Re}\, c_j|^2 \leq (\text{Im}\, c_j)^2$, for some $j$, so we obtain

**Lemma 1.** (Jensen's circles) *If* $f \in LP^*$*, then non-real zeros of* $f'$ *belong to the union of discs* $|z - \text{Re}\, c| \leq |\text{Im}\, c|$ *over all non-real zeros* $c$ *of* $f$*.*

Let us prove Theorem 1 by contradiction. Suppose that $f \in LP^*$, and all $f^{(n)}$ have some non-real zeros. The number of non-real zeros is a non-increasing as a function of $n$. This follows from the similar fact for polynomials, which is a consequence of Rolle's theorem.

Now it is easy to see that there exists an infinite sequence $(z_n)$ in the upper half-plane $H$ such that $f^{(n)}(z_n) = 0$, and the Jensen condition is satisfied:

$$|z_{n+1} - \operatorname{Re} z_n| \leq (\operatorname{Im} z_n)^2 \tag{9}$$

Indeed, by Lemma 1 there are finite sequences of any length with this property. Taking subsequences and then applying the diagonal procedure, we obtain an infinite sequence.

Now we use Lemma 1 to estimate the rate of growth of $z_n$, more precisely we need

$$S_{m,n} = |z_m - z_{m+1}| + |z_{m+1} - z_{m+2}| + \ldots + |z_{m+n-1} - z_{m+n}|.$$

Using the Cauchy–Schwarz inequality and (9) we obtain

$$S_{m,n}^2 \leq n \sum_{k=m}^{m+n-1} |z_k - z_{k+1}|^2 \leq 2n \sum_{k=m}^{m+n-1} \beta_k(\beta_k - \beta_{k+1}),$$

where $\beta_k = \operatorname{Im} z_k$. It follows from (9) that $\beta_k$ is a decreasing sequence. Let $\beta = \lim \beta_k \geq 0$. Then we obtain

$$S_{m,n} \leq \sqrt{2n(\beta_m(\beta_m - \beta))},$$

so for every $\epsilon > 0$ there exists $m_0$ such that $S_{m,n}/\sqrt{n} < \epsilon$ for $m \geq m_0$. This implies that

$$S_{0,n} = o(\sqrt{n}), \quad n \to \infty. \tag{10}$$

Now we estimate our function $f$ using a result of Goncharov. For an entire function $f$ we use the standard notation

$$M(r, f) = \max_{|z| \leq r} |f(z)|.$$

**Lemma 2.** (Goncharov's inequality) *Let $f$ be an entire function and $f^{(n)}(z_n) = 0$ for some sequence $(z_n), n = 0, 1 \ldots.$ Then for every $n \geq 1$ we have*

$$|f(z)| \leq \frac{M_n}{n!}(|z - z_0| + |z_0 - z_1| + \ldots + |z_{n-2} - z_{n-1}|)^n, \quad z \in \mathbf{C},$$

*where*

$$M_n = M(|z| + |z - z_0| + |z_0 - z_1| + \ldots + |z_{n-2} - z_{n-1}|,\ f^{(n)}).$$

*Proof.* By the Newton-Leibniz formula,

$$f(z) = \int_{z_0}^{z} \int_{z_1}^{\zeta_1} \dots \int_{z_{n-1}}^{\zeta_{n-1}} f^{(n)}(\zeta_n) d\zeta_n d\zeta_{n-1} \dots d\zeta_2 d\zeta_1. \tag{11}$$

To estimate the integral, we consider the broken line $L$ connecting $z_{n-1}, z_{n-2}, \dots, z_0, z$. This broken line evidently belongs to the disk of radius

$$|z| + |z - z_0| + \dots + |z_{n-2} - z_{n-1}|$$

centered at the origin. Thus $M_n$ is the upper estimate for $|f^{(n)}|$ on $L$. We choose the path of integration in all integrals in (11) to be parts of $L$. Set

$$t_j = |z_{n-1} - z_{n-2}| + |z_{n-2} - z_{n-3}| + \dots + |z_{j+1} - z_j|, \quad 0 \le j \le n-2, \quad t_{n-1} = 0.$$

Then put $s = t_0 + |z - z_0|$ and let $s_k$ be the length of the part of $L$ from $z_{n-1}$ to $\zeta_k$. With these notation,

$$|f(z)| \le M_n \int_{t_0}^{s} ds_1 \int_{t_1}^{s_1} ds_2 \dots \int_{t_{n-1}}^{s_{n-1}} ds_n.$$

So we have

$$t_0 \le s_1 \le s, \ t_1 \le s_2 \le s_1, \dots, t_{n-1} \le s_n \le s_{n-1},$$

and $t_{j+1} \le t_j, \ 0 \le j \le n-2$. Replacing all lower limits by 0, we obtain

$$|f(z)| \le M_n \int_0^{s} ds_1 \int_0^{s_1} ds_2 \dots \int_0^{s_{n-1}} ds_n \le M_n s^n / n!.$$

Our function $f \in LP^*$ has at most order 2 normal type, which means

$$|f(z)| \le C e^{a|z|^2}$$

with some $a, C > 0$. We use Cauchy's inequalities to estimate $f^{(n)}$:

$$M(\sqrt{n}, f^{(n)}) \le n!(r - \sqrt{n})^{-n} M(r, f) \le C n!(r - \sqrt{n})^{-n} e^{ar^2},$$

where $r > \sqrt{n}$. Then we minimize the RHS for $r > \sqrt{n}$. The result is

$$\left( \frac{M(\sqrt{n}, f^{(n)})}{n!} \right)^{1/n} = O\left( \frac{1}{\sqrt{n}} \right), \quad n \to \infty. \tag{12}$$

Combining this with Lemma 2 and (10), we obtain for any fixed $z \in H$ and every $\epsilon > 0$

$$|f(z)| \leq (\epsilon\sqrt{n})^n \frac{M_n}{n!} \leq (C\epsilon)^n,$$

where $C$ is independent of $n$. This shows that $f(z) = 0$ and proves the theorem.

## 4. Functions with positive imaginary part in $H$

Class $\mathfrak{R}$ frequently occurs in a great variety of questions, especially in spectral theory. It is known under the names Nevanlinna class, Herglotz class or Krein's class. The parametric representation of the whole class is the following:

$$\phi(z) = \lambda z + b - \int_{-\infty}^{\infty} \left( \frac{1}{z-t} + \frac{t}{1+t^2} \right) d\mu(t), \tag{13}$$

where $\lambda \geq 0$, $b \in \mathbf{R}$ and $\mu$ is a non-negative measure with the property

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty.$$

We will also use the following inequalities for $\psi \in \mathfrak{R} \backslash \{0\}$:

$$|\psi(i)| \frac{\operatorname{Im} z}{(1+|z|)^2} \leq |\psi(z)| \leq |\psi(i)| \frac{(1+|z|)^2}{\operatorname{Im} z}, \tag{14}$$

$$\left| \frac{\psi'(z)}{\psi(z)} \right| \leq \frac{1}{\operatorname{Im} z}, \tag{15}$$

$$\left| \log \frac{\psi(z+\zeta)}{\psi(z)} \right| \leq 1, \quad |\zeta| < \frac{1}{2} \operatorname{Im} z. \tag{16}$$

All these inequalities are consequences of the Schwarz lemma, which says that a holomorphic map of a disk into itself does not increase the hyperbolic metric (which means exactly (15).

**Lemma.** (Angular derivative) *There exists* $\lambda \geq 0$ *such that*

$$\psi(z) = \lambda z + \psi_1(z),$$

*where* $\psi_1 \in \mathfrak{R}$ *and* $\psi_1(z) = o(z)$ *in every Stolz angle.*

This can be easily deduced from the representation (13).

**5. Wiman's scale.**

In the rest of these lectures we discuss real entire functions with almost all zeros real, which do not belong to the class $LP^*$. For such functions of finite order, A. Wiman introduced a classification which is more refined than the usual classification by genus. For every non-negative integer $p$, the class class $V_{2p}$ consists of all real entire functions of the form

$$f(z) = e^{az^{2p+2}} w(z), \tag{17}$$

where $a \leq 0$, and $w$ is a real entire function of genus at most $2p+1$ with all zeros real, that is

$$w(z) = z^m e^{Q(z)} \prod_k \left(1 - \frac{z}{z_k}\right) \exp\left\{\frac{z}{z_k} + \ldots + \frac{1}{2p+1}\left(\frac{z}{z_k}\right)^{2p+1}\right\},$$

where $Q$ is a real polynomial of degree at most $2p+1$, and $(z_k)$ is a real sequence, finite or infinite.

**Definition.** $W_{2p} = V_{2p} \backslash V_{2p-2},\ p \geq 1$, and $W_0 = V_0$.

So $W_0 = LP$. The union of disjoint sets $W_{2p},\ p \geq 0$ consists of all real entire functions of finite order with all zeros real.

We also define $W_{2p}^*$ as the set of products $f = Ph$ where $h \in W_{2p}$ and $P$ is a real polynomial without real zeros. The degree of this polynomial $P$ is $N(f)$.

What happens with $N(f)$ when we differentiate?

**Theorem.** (Laguerre, Borel) *If $f \in W_{2p}^*$, then $N(f') \leq N(f) + 2p$.*

This will be proved together with the following

**Fundamental Lemma.** *Let $h \in W_{2p}$. Then*

$$h'/h = P_0 \psi_0, \tag{18}$$

*where $P_0$ is a real polynomial,* $\deg P_0 = 2p$*, the leading coefficient of $P_0$ is negative, and $\psi_0 \in \mathfrak{R} \backslash \{0\}$.*

This factorization was stated by Levin and Ostrovskii for arbitrary entire functions with almost all zeros real, and $P_0$ a real entire function. This factorization was used in all subsequent work on the subject. In the case $h \in W_{2p}^*$, we have a similar factorization with a polynomial $P_0$ of degree at most $2p + N(f)$, which follows from the Laguerre–Borel theorem. Exact statement as above is due to Hellerstein and Williamson.

*Proof of the Fundamental Lemma and of the Theorem of Laguerre-Borel.* Let $f \in W_{2p}^*$, $f = Ph$, where $P$ is a real polynomial without real roots.

We write $f = \lim f_n$, where

$$f_n(z) = P(z)e^{Q(z)} \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right) e^{Q_k(z)},$$

where $\deg Q_k \leq 2p+1$ and $\deg Q \leq 2p+2$, when $\deg Q = 2p+2$ the leading coefficient of $Q$ is negative.

$$\frac{f_n'(z)}{f_n(z)} = \frac{P'(z)}{P(z)} + Q'(z) + \sum_{k=1}^{n} \left( Q_k'(z) + \sum_{k=1}^{n} \frac{1}{z - z_k} \right).$$

These logarithmic derivatives are $O(z^{2p+1})$ on the imaginary axis, and $O(z^{2p})$ when $\deg Q \leq 2p + 1$.

By Rolle's theorem, there is an odd number of zeros of $f_n'$ on any interval between the zeros of $f_n$. We choose one on each interval and designate it as a "Rolle's zero". Moreover, if $\deg Q = 2p + 2$, the leading coefficient of $Q$ is negative, then $f_n(z) \to 0$ as $z \to \infty$ and we have two extra zeros of $f_n$ outside the convex hall of real zeros of $f_n$. We also count them as Rolle's zeros in this case. We order all these Rolle's zeros in an increasing sequence: $(w_k)$.

Using alternating sequences $(z_k)$ and $(w_k)$ we form the rational product $\phi_n(z)$ with poles $z_k$ and zeros $w_k$. As poles and zeros alternate, $\operatorname{Im} \phi_n$ has constant sign in $H$. We put a multiple $\pm 1$ in front of $\phi_n$, to ensure that $\phi_n \in \mathfrak{R}$. Then we have

$$f_n'/f_n = S_n \phi_n,$$

where $S_n$ is a rational function whose poles are zeros of $P$, and zeros are "extraordinary zeros" of $f_n'/f_n$, that is those which are not Rolle's zeros. Notice that the number of extraordinary zeros is always even.

Growth comparison on the imaginary axis shows that $S_n$ has a pole at infinity of order at most $2p + 1$ if $\deg Q \leq 2p + 1$ and at most $2p$ if $\deg Q = 2p + 2$. As the order of $S$ at infinity must be even, we conclude that in all cases $S_n(z) = O(z^{2p})$. This the number of extraordinary zeros does not exceed $2p + \deg P$.

This proves the Laguerre-Borel theorem.

To finish the proof of the Fundamental lemma, we assume that $P = 1$, $f = h$, so $P_0 = \lim S_n$ is a polynomial of degree at most $2p$, and we have the representation (18) with $\psi_0 = \lim \phi_n$.

First we show that the leading coefficient of $P_0$ is negative. Let $a_0$ be a zero of $h$. Then the residue of $h'/h$ at $a_0$ is positive, and the residue of $\psi_0$ is negative, so $P_0(a_0) < 0$. Since the number of extraordinary zeros on each interval $(a_j, a_{j+1})$ is even, the total number of real extraordinary zeros on the left of $a_0$ is even. So $P_0(x) \to -\infty$ as $x \to -\infty$. As $P_0$ is of even degree, we conclude that its leading coefficient is negative.

Now we prove that $d = \deg P_0 = 2p$, so

$$P_0(z) = cz^d + \ldots, \quad c < 0.$$

Let $\lambda_0 \geq 0$ be the angular derivative of $\psi_0$. Then in every Stolz angle

$$h'(z)/h(z) = c\lambda_0 z^{d+1} + o(z^{d+1}), \quad z \to \infty.$$

Integrating this along the straight lines we obtain

$$\log h(z) = \frac{c\lambda_0}{d+2} z^{d+2} + o(z^{d+2}), \quad z \to \infty.$$

If $c\lambda_0 < 0$, we compare this with (17) and obtain that $d = 2p$. If $\lambda_0 = 0$ we obtain that $a = 0$ in (17), so the genus $g$ of $h$ is at most $2p + 1$. The logarithmic derivative of $h$ has the form

$$\frac{h'(z)}{h(z)} = T(z) + z^g \sum_j \sum_j \frac{m_j}{a_j^g (z - a_j)}, \tag{19}$$

where $T$ is a polynomial. On the other hand (18) combined with (4) gives

$$\frac{h'(z)}{h(z)} = P_0(z) \left( \lambda_0 z + b - \sum_j A_j \left( \frac{1}{z - a_j} + \frac{a_j}{1 + a_j^2} \right) \right), \tag{20}$$

where $\lambda_0 \geq 0$, $A_j \geq 0$ and $b$ is real. We also have

$$\sum_j \frac{A_j}{a_j^2} < \infty. \tag{21}$$

Equating the residues at the poles of (19) and (20), we obtain

$$P(a_j) = -m_j/A_j < 0. \tag{22}$$

Now we have $0 < -P_0(a_j) \leq C|a_j|^d$ for some $C > 0$. Then (21) and (22) imply

$$\frac{1}{C}\sum_j \frac{m_j}{|a_j|^{d+2}} \leq -\sum_j \frac{m_j}{a_j^2 P_0(a_j)} = \sum_j \frac{A_j}{a_j^2} < \infty,$$

which shows that $d+2 \geq g+1$, that is $d \geq g-1$. If $g = 2p+1$ we conclude that $d \geq 2p$. If $g = 2p$ we conclude the same because $d$ is even. It remains to notice that $g \geq 2p-1$ because $h$ belongs to $W_{2p}$. This completes the proof of the lemma.

**Remark.** Class $\mathfrak{R}$ which is also denoted $\mathbf{N}_0$ in honor of Nevanlinna can be characterized by the following property: it consists of functions analytic in $\mathbf{C}\backslash\mathbf{R}$ for which all quadratic forms

$$\frac{\phi(z_j) - \phi(\overline{\zeta_k})}{z_j - \overline{\zeta_k}} w_j \overline{w_k}$$

are positive semi-definite. This is the Schwarz–Pick Theorem. Krein and Langer considered generalized Nevanlinna classes $\mathbf{N}_\kappa$ which consist of of functions for which these forms have at most $\kappa$ negative squares. Functions in the Fundamental Lemma belong to these classes $\mathbf{N}_\kappa$.

## 6. Beginning of the proof of Theorem 2 for functions of finite order. Rescaling and the saddle point asymptotics.

Let $f = Ph$, where $P$ is a real polynomial, and $h \in W_{2p},\ p \geq 1$. The Fundamental lemma gives

$$\frac{f'}{f} = P_0\psi_0 + \frac{P'}{P}, \quad \deg P_0 = 2p \geq 2. \tag{23}$$

Using (14) we obtain

$$\frac{rf'(ir)}{f(ir)} \to \infty, \quad r \to \infty. \tag{24}$$

Proving the theorem by contradiction, we assume that $N(f^{(k)})/k \to 0$ for $k \in \sigma,\ k \to \infty$, where $\sigma$ is some sequence. Using (24), we find positive numbers $a_k \to \infty$, such that

$$\left|\frac{a_k f'(ia_k)}{f(ia_k)}\right| = k,$$

and define

$$q_k(z) = \frac{a_k f'(a_k z)}{k f(a_k z)}.$$

Then $|q_k(i)| = 1$, and (23) with (14) imply that $\{q_k\}$ is a normal family. So passing to a subsequence we may assume that

$$q_k \to q, \quad k \to \infty$$

uniformly on compact subsets of $H$. We obtain from the Fundamental lemma that

$$q(z) = -z^{2p}\psi(z), \tag{25}$$

where $\psi : H \to \overline{H}\backslash\{0\}$ is a function of class $\mathfrak{R}$. We choose a branch of the log in a neighborhood of $f(ia_k)$, put $b_k = \log f(ia_k)$ and define

$$Q(z) = \int_i^z q(\zeta)d\zeta, \quad Q_k(z) = \int_i^z q_k(\zeta)d\zeta.$$

Then

$$Q_k(z) = \frac{\log f(a_k z) - b_k}{k} \to Q(z), \quad k \to \infty$$

uniformly on compacts in $H$, and $Q_k(i) = 0$. Our branches of log are well defined on every compact subset of $H$ when $k$ is large enough because $f$ has finitely many zeros in $H$ and $a_k \to \infty$.

Let $z$ be a point in $H$ and $0 < t < \operatorname{Im} z$. Then the disk $\{\zeta : |\zeta - a_k z| < t a_k\}$ is in $H$ and does not contain zeros of $f$ when $k$ is large. Cauchy's formula gives

$$\begin{aligned} f^{(k)}(a_k z) &= \frac{k!}{2\pi i}\int_{|\zeta|=a_k t} \frac{f(a_k z + \zeta)}{\zeta^k}\frac{d\zeta}{\zeta} \\ &= \frac{k!}{2\pi i}\int_{|\zeta|=a_k t} \frac{\exp(kQ_k(z+\zeta/a_k) + b_k)}{\zeta^k}\frac{d\zeta}{\zeta}. \end{aligned} \tag{26}$$

So

$$|f^{(k)}(a_k z)| \leq \frac{k!}{(a_k t)^k}\exp\left(\operatorname{Re} b_k + k\max_{|\zeta|=t}\operatorname{Re} Q_k(z+\zeta)\right),$$

and defining

$$u_k(z) = \frac{\log |f^{(k)}(a_k z)| - \operatorname{Re} b_k - \log k!}{k} + \log a_k,$$

we obtain

$$u_k \leq \max_{|\zeta|=t} \Re Q_k(z+\zeta) - \log t.$$

Since $Q_k \to Q$ we deduce that the $u_k$ are uniformly bounded from above on compact subsets, hence after choosing a subsequence, we obtain

$$u_k \to u \quad \text{in} \quad D'.$$

Here $u$ is a subharmonic function or $u = -\infty$.

The Riesz measure $\Delta u$ is the limit of the Riesz measures of $u_k$. So our assumption that $N(f^{(k)}) = o(k)$ will imply that $u$ is harmonic in $H$.

The plan is the following: we will derive a functional equation for $u$,

$$u\left(z - \frac{1}{q(z)}\right) = \operatorname{Re} Q(z) + \log |q(z)|, \quad z \in S, \tag{27}$$

where $S = \{a : |z| > R, \delta < \arg z < \pi - \delta\}$ for some positive $R, \delta < \pi/4$, using the saddle point asymptotics in (26), and then prove that this functional equation cannot have harmonic solutions. Saddle point asymptotics was used by Pólya to obtain the limit distribution of zeros of $f^{(k)}$ when $f = \exp P$, with a polynomial $P$. In this simple case the asymptotic distribution is obtained explicitly. Our new ingredients are rescaling $f(a_k z)$ and the study of the resulting functional equation for $u$. In the special case when $f$ is of completely regular growth, which means that $r^{-\rho} \log |f(rz)|$ has a limit in $D'$, as $r \to \infty$, such a functional equation was obtained by Evgrafov.

To explain the derivation of (27) we rewrite (26) as

$$f^{(k)}(a_k w) = \frac{k!}{2\pi i} \int_{|\zeta|=r_k} \exp\left(\log f(a_k(w+\zeta)) - k \log a_k \zeta\right) \frac{d\zeta}{\zeta},$$

which some $r_k > 0$ which we can choose. The saddle point method involves the stationary point of the function under the exponent, that is a solution of the equation

$$\frac{d}{d\zeta}\left(\frac{\log f(a_k(w+\zeta))}{k} - \log a_k \zeta\right) = \frac{a_k f'(a_k(w+\zeta))}{k f(a_k(w+\zeta))} - \frac{1}{\zeta} = q_k(w+\zeta) - \frac{1}{\zeta} = 0.$$

Instead of slowing this equation, we choose an arbitrary $z$ is a Stolz angle, and set $w = z - 1/q_k(z)$ and $\zeta = z - w = 1/q_k(z)$. With this choice, $\zeta$ is a saddle point, and we obtain an asymptotics by evaluating the function under the integral (28) at this point. This gives

$$\frac{1}{k}\left(\log f^{(k)}\left(a_k\left(z - \frac{1}{q_k(z)}\right)\right) - c_k\right) \sim Q_k(z) + \log q_k(z).$$

Letting $k \to \infty$ we obtain (27).

Of course, this asymptotics needs justification. This is obtained using the estimates which $f'/f$ satisfies due to the Fundamental Lemma.

After that it remains to prove that there is no harmonic function that satisfies (27)

## 7. Functional equation and completion of the proof.

We recall that

$$q(z) = -z^{2p}\psi(z),$$

where $\phi : H \to \overline{H}\backslash\{0\}$,

$$\psi(z) = \lim_{k\to\infty} \frac{\psi_0(a_k z)}{|\psi_0(ia_k)|}.$$

**Lemma.** *Define*

$$Q(z) = \int_i^z q(\zeta)d\zeta,$$

*and*

$$F(z) = z - \frac{1}{q(z)}. \tag{28}$$

*Then there is no harmonic function $u$ in $H$ satisfying*

$$u(F(z)) = \operatorname{Re} Q(z) + \log|q(z)|$$

*in a Stolz angle.*

*Proof.* Suppose that there is such a harmonic function Then there exists a holomorphic $h$, $u = \operatorname{Re} h$ and

$$h(F(z)) = Q(z) + \log q(z).$$

Differentiating and using $q = Q'$, and using (28), we obtain

$$h'(F(z)) = q(z) = \frac{1}{z - F(z)}.$$

It follows that there exists a branch $G$ of the inverse $F^{-1}$ which is defined in a Stolz angle $S$ and satisfies

$$G(w) \sim w, \quad w \to \infty, w \in S. \tag{29}$$

In particular, $G(w) \in H$ for $w \in S$ and $w$ large enough. We have

$$h'(w) = q(G(w)) = \frac{1}{G(w) - w} \tag{30}$$

for $w \in S$ and $|w|$ large enough. Since $h$ is holomorphic in $H$, we see that $G$ has a meromorphic continuation to $H$. Using the definition of $q$, second equation in (30) can be rewritten as

$$\psi(G(w)) = \frac{1}{G^{2p}(w)(w - G(w))}. \tag{31}$$

We will derive from this that $G$ maps $H$ into itself. It is sufficient to show that $\operatorname{Im} G(z) \neq 0$ in $H$. In view of (29) there exists a point $w_0 \in H$ such that $G(w_0) \in H$. If $g$ takes a real value in $H$, then there exists a curve $\phi : [0,1] \to H$ beginning at $w_0$ and ending at some point $w_1$, Such that $G(w(t)) \in H$ for $0 \leq t < 1$, but $G(\phi(1)) \in \mathbf{R}$. We may assume that $G(w_1) \neq 0$; this can be achieved by a small perturbation of the curve $\psi$ and the point $w_1$. Using (31) we obtain an analytic continuation of $\psi$ to the real point $G(w_1)$ along the curve $G(\phi)$. We have

$$\lim_{t \to 1} \operatorname{Im} \psi(G(\phi(t))) \geq 0,$$

because the imaginary part of $\psi$ is non-negative in $H$. It follows that as $w \to w_1$, the RHS of (31) has negative imaginary part, while the LHS has non-negative imaginary part, which is a contradiction. So $G$ maps $H$ into itself.

Then Lemma on the angular derivative and (29) shows that $\operatorname{Im}(G(w) - w) > 0$ in $H$. Combining this with the second equation (30) we obtain

$\text{Im}\, q(G(w)) < 0$ for $w \in H$. Using (29) we find in particular that $\text{Im}\, q(e^{i\pi/(2p)}y) <$ $0$ for large $y$. On the other hand,

$$\text{Im}\, q(e^{i\pi/(2p}y) = y^{2p}\text{Im}\, \phi(e^{i\pi/(2p)}y) \geq 0.$$

This contradiction proves the Lemma.

A simple compactness argument shows that there exist positive $c$ and $\delta$ depending only on $p$ such that the subharmonic function $u$ satisfying (27) must have Riesz measure at least $c$ in $\delta < \arg z < \pi - \delta$. However no explicit estimate of these $\delta$ and $c$ is known.

## Some ingredients of the proof of the Wiman conjecture

### 8. Sheil-Small's argument

We recall the result.

*Let $f$ be a real entire function with all zeros real. Then $f''$ has some non-real zeros.*

The condition that $f$ is real is essential: for $f(z) = e^{e^{iz}}$, $f''$ has only real zeros. Also, second derivative cannot be replaced by the first: for $f(z) = e^{\sin z}$, $f'$ has all zeros real but $f \notin LP$.

The simplest special case is that $f$ is of finite order, and has no zeros. Even this special case was unsolved for long time.

So we have $f = e^p$, $f'' = (p'' + p^2)e^p$, and setting $L = p' = f'/f$ we have to prove the following

**Theorem.** (Sheil-Small, 1989). *For a real polynomial $L$ of degree $\geq 2$, zeros of $L' + L^2$ cannot be all real.*

If all roots of $L$ are real, this was conjectured by Pólya in 1917 and proved by Prüfer in 1918. Here is Prüfer's proof. Let $x_1 < \ldots < x_k$ be all distinct roots of $L$. It is easy to see that the number of common real roots of $L$ and $L' + L^2$ is $n - k$, where $n = \deg L$. A root of $L' + L^2$ which is not a root of $L$ satisfies $L'/L^2 = -1$. But the function $L'/L^2$ is strictly monotone on every interval between the adjacent toots of $L$:

$$L\frac{d}{dx}\left(\frac{L'}{L^2}\right) = \left(\frac{L'}{L}\right)' - \left(\frac{L'}{L}\right)^2 < 0.$$

This proves that $L'+L^2$ has at most $n+1$ real roots if all roots of $L$ are real.

The following ingenuous proof of the general case due to Sheil–Small. Notice that it uses complex variables, unlike Prüfer's proof.

Let $F(z) = z - 1/L$. Consider the sets

$$\Lambda = \{z \in H : \operatorname{Im} L > 0\} \quad \text{and} \quad K = \{z \in H : \operatorname{Im} F(z) > 0\}.$$

As $\operatorname{Im} L > 0$ implies $\operatorname{Im}(z - 1/L(z)) > 0$, every component $V$ of $\Lambda$ is contained in a component $U$ of $K$.

Notice that all components of $\Lambda$ are unbounded, by the maximum principle.

A zero $\zeta$ of $L$ is called *good* if $\zeta$ is real and $L'(\zeta) < 0$. All other zeros of $L$ are called *bad*. Each bad zero lies on $\partial V \cap \partial U$ for some components $V \subset U$ of $\Lambda$ and $K$.

If $U$ is a component of $K$, then $G : U \to H$ is a covering, and the degree of this covering must be equal to the number of bad zeros on $\partial U$ plus one, if $U$ contains some $V$ because components of $V$ are unbounded. Thus such a component $U$ must contain critical points of $G$. But critical points of $G$ in $K$ are zeros of $L' + L^2$:

$$(z - 1/L)' = 1 + L'/L^2 = (L^2 + L')/L^2.$$

It remains to notice that $L$ has no bad zeros only in the case when $L$ has only one good zero, that is $L(z) = -az + b$.

One can improve this argument by saying that the number of zeros of $L' + L$ in $H$ is at least the number of bad zeros. A more careful count shows that multiplicity can be also taken into account.

We obtain:

**Exercise.** *The number of non-real roots of* $L' + L^2$ *is at least* $\deg L - 1$*, and the number of real roots is at most* $\deg L + 1$*.*

The proof of Sheil-Small extends to the case of finite order $f$ using the Fundamental Lemma. Now $L$ is the logarithmic derivative which has a representation of the Fundamental Lemma. It can have simple poles, but these poles are real, and residues at them are positive. It follows that no pole can lie on $\partial\Lambda$. So we have that the components of $\Lambda$ are unbounded.

Applying the representation of the Fundamental Lemma to $e^{-cz} f(z)$ with a real $c$, we obtain that $L - c$ is a product of a polynomial with a function with positive real part in $H$. Therefore $L$ takes every real value finitely many times in $H$ (at most $p$).

Now consider the boundary $C$ of a component $V$ of $\lambda$. It is mapped by $L$ into the real line monotonically, so $C$ is a curve unbounded in both directions, and the image of this curve is an interval $(a, b) \subset \mathbf{R}$. It follows from the Lindelöf theorem that $a = b = \infty$, and we conclude that there are finitely many components $V$, each bounded by finitely many curves $C$.

Each component $V$ is contained in some component $U$ of $K$. Furthermore $F$ has no critical points in $H$. If there is at least one bad zero on $\partial U$, then $F : U \to H$ cannot be bijective, so $F|_U$ must have an asymptotic value in $H$.

This means that there is a curve $\gamma : (0, 1) \to U$ which tends to $\infty$ and such that $F(\gamma(t)) \to \alpha \in H$ as $t \to 1$. Then we have

$$L(z) \to 0, \tag{32}$$

and $zL(z) \to 1$ on $\gamma$. Let $z = x + iy \in \gamma$, $|z| = r$. By (14) we have $|\psi(z)| > Ay/r^2$ with some $A > 0$. Also $\phi(z) > Ar^2$. Since $zL(z) \to 1$ on $\gamma$, we conclude that $y \to 0$ on $\gamma$. Now since $F(z) \to \alpha = \delta + i\epsilon$ on $\gamma$, where $\epsilon > 0$, we see taking imaginary parts that $y - \operatorname{Im} 1/L(x) \to \epsilon$, so $-\operatorname{Im} 1/L(z) \to \epsilon$. Thus for large $|z|$, $\operatorname{Im} L(z) > 0$ on $\gamma$ so $z \in \Lambda$ and we obtain a contradiction between (32) and $L(z) \to \infty$ in $\Lambda$.

**Exercise.** Prove that if $f = Ph$, where $P$ is a real polynomial, and $h \in W_{2p}$, then $f''$ has at least $2p$ non-real zeros.

The most complicated case is that of infinite order. For this case, we only outline some main ideas. When $f$ has infinite order, the Fundamental Lemma only says that

$$L = \phi\psi, \tag{33}$$

where $\psi \in \mathfrak{R}\backslash\{0\}$, and $\phi$ is an arbitrary real entire function. The main part of Sheil-small argument gives the following.

**Proposition.** *Let*

$$F(z) = z - 1/L(z),$$

*where L is defined by (33). Assume that $\psi$ in (33)has at least one zero. Then $F$ has at least one non-real asymptotic value in $H$. This means that there is a curve $\gamma$ in $H$ tending to infinity, and such that $F(z) \to \alpha \in H$ as $z \to \infty$, $z \in \gamma$.*

*Proof.* First of all $F$ has no critical values over $H$, that is no zeros $z$ of $F'$ such that $H(z) \in H$. This is because

$$F' = 1 + L'/L^a = f''f/(f')^2,$$

and $f$ and $f''$ have only real zeros.

Second, all components $C$ of the set $\{z \in H : L(z) \in H\}$ are unbounded, and

$$\limsup_{z\to\infty, z\in C} \operatorname{Im} L(z) > 0 \tag{34}$$

for each such component. This is because a pole of $L$ cannot lie on the boundary of such a component, by the same argument as above. Then by assumption there is at least one zero of $\phi$, which is a zero of $L$. Then $L$ must have at least one extraordinary zero (either lying in $H$ or a real zero $x$ with $L'(x) > 0$. This is because the number of extraordinary zeros on every interval between two poles of $L$ is even. This extraordinary zero must be on the boundary of some component $V$ of $\Lambda$. Then there is a component $U$ of $K$ which contains $V$ and has the3 same zero on the boundary. If the restriction $F : U \to H$ is a covering, then the degree of this covering is at least 2, because $F$ has at least one pole on $\partial U$, and in addition $\limsup F(z) = \infty$ when $z \to \infty$ in $U$, which follows from (34). But $F$ has no critical points, therefore it must have an asymptotic value.

The rest of the proof of the Wiman conjecture consist of estimates of $F$ which eventually lead to the conclusion that $F$ cannot have an asymptotic value.

An arbitrary entire function $\phi$ can behave at $\infty$ in an arbitrarily complicated way. To make some conclusion one needs a growth estimate of $\phi$. This is performed with the help of a proper generalization of Nevanlinna theory.

## 9. Nevanlinna theory, Hayman's theorem and a lemma of Zalcman-Pang.

Everyone knows Picard's theorem: *if a meromorphic function $f$ omits* 3 *values than $f$ is constant.*

Nevanlinna theory gives a quantitative version of this. Let $n(r, f)$ be the counting function of poles in $\{z : |z| \leq r\}$ (counting multiplicity), and

$$N(r, f) = \int_0^r (n(t, f) - n(0, f))\frac{dt}{t} + n(0, f)\log r,$$

$$m(r,f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta.$$

The Nevanlinna characteristic is

$$T(r,f)=N(r,f)+m(r,f).$$

This is an increasing function of $r$, and for rational functions we have $T(r,f)=d\log r+O(1)$, where $d$ is the degree, while for transcendental functions $T(r,f)/\log r\to\infty$.

The Nevanlinna characteristic should be considered as a generalization of degree of a rational function to transcendental functions. It enjoys the following properties:

$$\begin{aligned}
T(r,f+g) &\leq T(r,f)+T(r,g)+O(1), && (35)\\
T(r,fg) &\leq T(r,f)+T(r,g)+O(1), && (36)\\
T(r,f^n) &= nT(r,f), && (37)\\
T(r,1/f) &= T(r,f)+O(1). && (38)
\end{aligned}$$

Notice that the last equality is Jensen's formula. All the rest are trivial.

**Exercise.** *If $T$ is a function from the set of rational functions to the set of positive numbers which satisfies these properties without $O(1)$ terms then $T(f)=c\deg f$, for some $c>0$.*

The main analytic fact is called the Lemma on the logarithmic derivative:

$$m(r,f'/f)=O(\log T(r,f)+\log^+ r),\quad r\to\infty, r\notin E,$$

where $E$ is a set of finite measure. We will denote any function like in the RHS of this formula by $S(r,f)$. This is the "error term" in Nevanlinna theory. A variant of the lemma on the logarithmic derivative for functions in the unit disk is

$$m(r,f'/f)=O(\log T(r,f)+\log(1-r)^{-1}),\quad r\to 1, r\notin E,$$

where $\int_E(1-r)^{-1}dr<\infty$.

Let $n_1(r,f)$ be the counting function of critical points of $f$ (including multiplicity), the explicit expression is

$$n_1(r,f)=n(r,1/f')+2n(r,f)-n(r,f').$$

We define $N_1(r,f)$ as before. Finally, let $\mathfrak{N}(r,f)$ be the Nevanlinna counting function of poles without multiplicity.

With these notation, the Second Main theorem of Nevanlinna says that

$$\sum_{j=1}^{q} m(r,(f-a_j)^{-1}) + N_1(r,f) \leq 2T(r,f) + S(r,f). \tag{39}$$

It follows that

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} \overline{N}(r,(f-a)^{-1}) + S(r,f).$$

Taking $q=3$, we see that this indeed a generalized Picard's theorem.

Nevanlinna theory permits to obtain many results similar to Picard's theorem, where instead of omitted values of $f$ one considers omitted values of derivatives.

For example,

**Theorem**. (Milloux) *If $f$ is a meromorphic function, and $f'$ is not constant, then*

$$T(r,f) \leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) - N\left(r,\frac{f'-1}{f''}\right) + S(r,f),$$

*Proof.* Applying the second main theorem to $f'$ we obtain

$$\begin{aligned} & m(r,1/f') + m(r,f') + m(r,1/(f'-1)) \leq 2T(r,f') - N_1(r,f') + S(r,f) \\ = \;& m(r,f') + m\left(r,\frac{1}{f'-1}\right) + N(r,f') + N\left(r,\frac{1}{f'-1}\right) \\ - \;& N\left(r,\frac{1}{f''}\right) - 2N(r,f') + N(r,f'') + S(r,f) \end{aligned}$$

Now $N(r,f'') - N(r,f') = \overline{N}(r,f)$, and

$$N\left(r,\frac{1}{f'-1}\right) - N(r,1/f'') = \overline{N}\left(r,\frac{1}{f'-1}\right) - N\left(r,\frac{f'-1}{f''}\right),$$

and we obtain

$$m(r,1/f') \leq \overline{N}(r,f) + N\left(r,\frac{1}{f'-1}\right) - N\left(r,\frac{f'-1}{f''}\right) + S(r,f). \tag{40}$$

Now

$$\begin{aligned} T(r,f) &\le m(r,1/f)+N(r,1/f)+O(1) \\ &\le m(r,1/f')+N(r,1/f)+m(r,f'/f)+O(1) \\ &\le m(r,1/f')+N(r,1/f)+S(r,f). \end{aligned}$$

Combining this with (40), we obtain the statement.

As a corollary one obtains that if $f$ omits 0 and $\infty$, and $f'$ omits 1, then $f$ is constant.

A remarkable improvement of this is the following

**Theorem**. (Hayman) *If $f$ is a meromorphic function, and $f'$ is not constant, then*

$$T(r,f) \le 3N(r,1/f)+4\overline{N}(r,(f'-1)^{-1})+S(r,f).$$

*Proof.* Consider the auxiliary function

$$g=\frac{(f'')^2}{(f'-1)^3}.$$

If $f(z)$ has a simple pole, then $g(z)$ is finite and $g'(z)=0$. Denoting by $N_s(r,f)$ the counting function of simple poles, we obtain

$$N_s(r,f) \le N(r,g/g'). \tag{41}$$

Function $g$ has poles at the same points where $f'-1=0$, so

$$\overline{N}(r,g) \le \overline{N}\left(r,\frac{1}{f'-1}\right). \tag{42}$$

The zeros of $g$ occur at multiple poles of $f$ and at the poles of $(f'-1)/f''$. Therefore

$$\overline{N}(r,1/g)=\overline{N}(r,f)-N_s(r,f)+\overline{N}(r,(f'-1)/f''). \tag{43}$$

From (42) and (43) we obtain

$$\begin{aligned} N(r,g'/g) &= \overline{N}(r,g)+\overline{N}(r,1/g) \\ &= \overline{N}\left(r,\frac{1}{f'-1}\right)+\overline{N}(r,f)-N_s(r,f)+\overline{N}\left(r,\frac{f'-1}{f''}\right). \end{aligned} \tag{44}$$

Applying to (41) the first main theorem, the lemma on the logarithmic derivative, and (44), we obtain

$$\begin{aligned} N_s(r,f) &\leq N(r,g'/g) + m(r,g'/g) - m(r,g/g') + O(1) \\ &\leq N(r,g'/g) + S(r,f) \\ &= \overline{N}\left(r,\frac{1}{f'-1}\right) + \overline{N}(r,f) - N_s(r,f) + \overline{N}\left(r,\frac{f'-1}{f''}\right), \end{aligned}$$

from which follows

$$2N_s(r,f) \leq \overline{N}\left(r,\frac{1}{f'-1}\right) + \overline{N}(r,f) + \overline{N}\left(r,\frac{f'-1}{f''}\right) \tag{45}$$

As multiple poles are counted in $N(r,f)$ at least twice, we obtain, using Milloux's theorem

$$\begin{aligned} & N_s(r,f) + 2(\overline{N}(r,f) - N_s(r,f)) \leq N(r,f) \leq T(r,f) \\ \leq\ & \overline{N}(r,f) + N(r,1/f) + \overline{N}\left(r,\frac{1}{f'-1}\right) - N\left(r,\frac{f'-1}{f''}\right) + S(r,f). \end{aligned}$$

Multiplying this by 2 and combining with (45), we obtain

$$\overline{N}(r,f) \leq 2N(r,1/f) + 3\overline{N}(r,1/(f'-1)) + S(r,f),$$

Combining this with Milloux inequality, we obtain the result.

It follows that $f(z) \neq 0$ and $f'(z) \neq 1$ imply that $f$ is constant. Unlike Picard's theorem, this only has two conditions on omitted values.

We will need two generalizations of this result of Hayman to functions in the upper half-plane.

First of them is valid for any region. We recall that Montel's theorem says that the family of meromorphic functions in any region which omit 3 fixed values, is normal. This is an instance of the general principle which is called the "Bloch Principle". If a condition imposed on a meromorphic function in the plane implies that the function is constant, then the same condition imposed on a family of meromorphic functions in a region must imply that this family is normal.

The version that we use is due to L. Zalcman and X. Pang.

**Theorem.** (Zalcman and Pang). *Let $F$ be a family of meromorphic functions in the unit disc without zeros, and suppose that $F$ is not normal. Then for every $\alpha > -1$ there exist a number $r \in (0,1)$, a sequence $z_n$, $|z_n| < r$, functions $f_n \in F$ and positive numbers $\rho_n \to 0$ such that*

$$\rho_n^{-\alpha} f_n(z_n + \rho_n z) \to f(z)$$

*uniformly (with respect to the spherical metric) on compact subsets of the plane, and the limit function $f$ is not constant.*

For example, taking $\alpha = 0$ we obtain that Montel's theorem follows from Picard's theorem.

*Proof.* We only give a proof for $\alpha = 1$, the case we need. If the family is not normal, there is a sequence $f_n$ and $z_n^*$, $|z_n^*| \leq r^* < 1$ such that

$$\frac{|f_n'(z_n^*)|}{1 + |f_n(z_n^*)|^2} \to \infty.$$

Choose arbitrary $r \in (r^*, 1)$. We have

$$\frac{(1 - |z_n^*/r|^2)^2 |f_n'(z_n^*)|}{(1 - |z_n^*/r|^2)^2 + |f_n(z_n^*)|^2} \geq \left(1 - \left|\frac{z_n^*}{r}\right|^2\right)^2 \frac{f_n'(z_n^*)|}{1 + |f_n(z_n^*)|^2}.$$

The RHS tends to $\infty$, so the left hand side also does. Therefore, we may assume that the LHS$\geq 1$.

Then we choose $t_n \in (0,1)$ and $z_n$, $|z_n| < r$ so that

$$\max_{|z|<r} \frac{(1 - |z/r|^2)^2 t_n^2 |f_n'(z)|}{(1 - |z/r|^2)^2 t_n^2 + |f_n(z)|^2} := \max_{|z|<r} \Phi_n(z) = 1,$$

and is achieved for $z = z_n$. Indeed, for $t_n = 1$, this expression is positive and $|z| < r$ and zero on the boundary (we used here the assumption that $f(z) \neq 0$). So there is a positive maximum, and this maximum tends to $+infty$. Then $t_n \to 0$, the maximum evidently tends to 0, so there exists $t_n$ for which this maximum equals 1.

Then we have

$$t_n^2 \frac{(1 - |z_n^*/r|^2)^2 |f_n'(z_n^*)|}{(1 - |z_n^*/r|^2)^2 + |f_n(z_n^*)|^2} \leq \Phi_n(z_n^*) \leq 1,$$

so $t_n \to 0$.

We set $\rho_n = (1-|z_n/r|^2)t_n$, then $\rho_n/(r-|z_n|) \to 0$, and thus the functions $h_n(z) = f_n(z_n + \rho_n z)/\rho_n$ are defined in the discs whose radii tend to $\infty$. It remains to estimate their spherical derivatives in these disks.

$$\frac{|h_n'(z)|}{1+|h_n(z)|^2} = \frac{\rho_n^2|f_n'(z_n+\rho_n z)|}{\rho_n^2+|f_n(z_n+\rho_n z)|^2} \leq 1,$$

while at $z = 0$ this is equal to 1. So $h_n$ tend to a limit $f$ and this limit is not constant.

**Corollary.** *Let $f$ be a function holomorphic in $H$, such that $f(z) \neq 0$ and $f''(z) \neq 0$. Then the family*

$$\{f(rz)/rf'(rz) : r > 0\}$$

*is a normal family.*

*Proof.* Let $g_r(z) = f(rz)/(rf'(rz))$. Then $g_r$ are meromorphic, zero-free, and

$$g_r' - 1 = -ff''/(f')^2,$$

and this is zero-free. So the family is normal by the previous theorem.

## 10. Tsuji characteristics

Second generalization of Hayman's theorem that we need is obtained by generalizing Nevanlinna theory to functions in a half-plane.

It was developed independently by Tsuji and Levin–Ostrovskii. They defined the characteristics in the following way. Let $\mathfrak{n}(r,f)$ be the number of poles, counting multiplicity is $\{z : |z-ir/2| \leq r/2, |z| \geq 1\}$, then

$$\mathfrak{N}(r,f) = \int_1^r \frac{n(t,f)}{t^2}dt, \quad r \geq 1,$$

and

$$\mathfrak{m}(r,f) = \frac{1}{2\pi}\int_{\sin^{-1}(1/r)}^{\pi-\sin^{-1}(1/r)} \frac{\log^+|f(re^{i\theta}\sin\theta)|}{r\sin^2\theta}d\theta,$$

The Tsuji characteristic is

$$\mathfrak{T}(r,f) = \mathfrak{m}(r,f) + \mathfrak{N}(r,f).$$

It is easy to show that the Tsuji characteristic has the same algebraic properties (35)-(38) as the Nevanlinna characteristic. The following version of the Lemma on the logarithmic derivative holds:

$$\mathfrak{m}(r,f'/f) = O(\log^+\mathfrak{T}(r,f) + \log r).$$

These formal properties permit to repeat all arguments of Milloux and Hayman in the upper half-plane, and to obtain the following

**Theorem.** (Levin–Ostrovskii). *If $g$ is meromorphic in $\overline{H}$, $g(z) \neq 0$ and $g'(z) \neq 1$, $z \in H$. Then $\mathfrak{T}(r,g) = O(\log r)$, $r \to \infty$.*

This is obtained by exactly the same arguments as Hayman's theorem but using the Tsuji characteristics instead of Nevanlinna characteristics.

In the Tsuji characteristics, the half-plane is exhausted by horocycles. We need to pass to the more usual exhaustion by concentric discs. For this the following lemma is used:

**Lemma A**. *Let $g$ be meromorphic in $\overline{H}$, and put*

$$m_{0\pi}(r,g) = \frac{1}{2\pi}\int_0^\pi \log^+ |g(re^{i\theta})| d\theta.$$

*Then for $R \geq 1$ we have*

$$\int_R^\infty \frac{m_{0\pi}(r,g)}{r^3} dr \leq \int_R^\infty \frac{\mathfrak{m}(r,g)}{r^2} dr.$$

This is proved by a change of the variables in the double integrals. It follows from this lemma that our entire function $\psi_0$ in the Levin–Ostrovskii representation is of order at most one.

A lower estimate will be obtained from the normal family argument:

**Lemma.** *For every $\delta > 0$ and $K > 1$ we have*

$$|wL(w)| > K, \quad \delta < \arg w < \pi - \delta, \tag{46}$$

*for all $|w|$ outside a set $E$ of zero logarithmic density.*

*Proof.* Consider the functions $g_r(z) = 1/(rL(rz)$ in a sector

$$\Omega = \{z : 1/2 < |z| < 2,\ \delta/4 < \arg z < \pi - \delta/4\}.$$

Suppose that $w_0$ is such that

$$|w_0 L(w_0)| > K, \quad |w_0| = r.$$

Then

$$|g_r(z_0)| \geq 1/K, \quad z_0 = w_0/r$$

and by normality

$$|g_r(z)| > 1/K_1, \quad |z| = 1, \delta/2 < \arg z < \pi - \delta_2,$$

with some $K_1 > 0$ which depends on $\delta$ and $K$, but is independent of $r$. This shows that

$$|wL(w)| = |w\psi_0(w)P_0(w)| \leq K_1$$

on most of the circle, which is incompatible with $P_0$ having order at most 1.

**Corollary.** Function $P_0$ has at least one zero.

## 11. Carleman's estimate. Direct and indirect singularities. Concluding argument.

*Let $u$ be a non-constant subharmonic function in the plane, $B(r,u) = \max_{|z|=r} u(z)$, and $\theta(r)$ the angular measure of the set*

$$\{\theta : u(re^{i\theta}) > 0\}.$$

*Suppose that $\theta(r) < 2\pi$ for all $r$, and $B(r_0, u) \geq 1$. Then*

$$\log \|u^+(4re^{i\theta})\|_1 \geq \log B(2r, u) \geq \pi \int_{r_0}^{r} \frac{dt}{t\theta(t)} + c,$$

*where $c$ is an absolute constant.*

This was proved by Carleman to derive what is called the Denjoy–Carleman–Ahlfors theorem: an entire function of order $\rho$ cannot have more than $2\rho$ asymptotic values. Our application is similar:

**Lemma.** *Function $F$ cannot have more than 4 finite non-real asymptotic values.*

*Proof.* Consider the auxiliary functions

$$G(z) = z^2 L(z) - z = \frac{zF(z)}{z} - F(z), \quad h_\alpha(z) = \frac{1}{F(z) - \alpha}$$

Then $g$ has no poles in $H$ and $\mathfrak{T}(r, g) + \mathfrak{T}(r, h) = O(\log r)$ by the properties of the Tsuji characteristic. Then Lemma A implies

$$\int_1^\infty \frac{m_{0\pi}(r, g) + m_{0\pi}(r, h_\alpha)}{r^3} dr < \infty. \tag{47}$$

Since $F$ is real on the real axis, we can take disjoint paths $\gamma_j$ in $H$ such that $F(z) \to \alpha_j$ $z \to \infty$. Let $D_j$ be domains between these paths, and $\theta_j(r)$ the angular measures of $D_j \cap \{z : |z| = r\}$. It is clear from the definition that $g(z) \to \alpha_j$ on $\gamma_j$, and as $g$ is holomorphic in $H$ it must grow in $D_j$. We apply Carleman's inequality to subharmonic functions

$$u_j(z) = \log^+ |g(z)/c|$$

. We obtain

$$\log \|u_j(4re^{i\theta})\|_1 \geq \pi \int^r \frac{dt}{t\theta_j(t)} + O(1),$$

so

$$\log m_{0\pi}(4r, g) \geq \int^r \frac{dt}{t\theta_j(t)} + O(1),$$

and by Cauchy–Schwarz inequality,

$$n^2 \leq \sum_{j=1}^{n} \theta_j(t) \sum_{j=1}^{n} \frac{1}{\theta_j(t)} \leq \pi \sum_{j=1}^{n} \frac{1}{\theta_j(t)}.$$

So $n \log r \leq \log m_{0\pi}(4r, g) + O(1)$ which contradicts (47).

So $F$ has finitely many asymptotic values and no critical values in $H$. As it has at least one asymptotic value, this must be a logarithmic singularity. Let $D$ be a component of preimage of a small disk around the asymptotic value $\alpha$ in the upper half-plane. Then $D$ contains no $\alpha$-points of $F$, and by Lemma B, $\theta(r) = o(1)$. Then Carleman's inequality applies to the subharmonic function

$$u(z) = \log^+ |h_\alpha/c|$$

shows that $h_\alpha$ is of infinite order, contradicting (47).

This completes the proof of the Wiman conjecture.

# References

[1] W. Bergweiler and A. Eremenko, Proof of a conjecture of Pólya on the zeros of successive derivatives of real entire functions, Acta math., 197 (2006) 145–166.

[2] W. Bergweiler, A. Eremenko and J. Langley, Real entire functions of infinite order and a conjecture of Wiman, Geom. Func. Anal 13 (2003) 975–991.

[3] A. Goldberg and I. Ostrovski, Value Distribution of meromorphic functions, AMS, Providence, RI, 2008.

[4] H. Ki and Y. Kim, On the number of non-real zeros of real entire functions and the Fourier–Pólya conjecture, Duke Math. J., 104 (2000) 45–73.

[5] J. Langley, Non-real zeros of higher derivatives of real entire functions of infinite order, J. Anal. Math., 97 (2005) 357–396.

[6] G. Pólya, Über die Nullstellen sukzessiver Derivierten, Math. Z., 12 (1922) 36–60.

[7] G. Pólya, On the zeros of the derivatives of a function and its analytic character, Bull. Amer. Math. Soc., 49 (1943) 178–191.

[8] T. Sheil-Small, On the zeros of the derivatives of real entire functions and Wiman's conjecture, Ann. Math., 129 (1989) 179–193.