Corona and cluster value problems in infinite-dimensional spaces

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Outline

- Introduction: Main Concepts
- Our Cluster Value Theorems
- Open Cluster Value Problems and Related Questions
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**Remark**

$\forall x^* \in X^*, x^* : B \rightarrow \mathbb{C}$ acts linearly and continuously, thus each $x^*$ is analytic and bounded.
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Examples of Banach algebras $H(B)$:

- $H^\infty(B)$: all bounded analytic functions on $B$.
- Two generalizations of the disk algebra:
  - $A_u(B)$: bounded and uniformly continuous analytic functions on $B$.
  - $A(B)$: uniform limits on $B$ of polynomials in the functions in $X^*$.
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One of the most important topics in the study of Banach algebras $H(B)$ is the study of its set of characters, the nonzero algebra homomorphisms from $H(B)$ to $\mathbb{C}$, called the spectrum of $H(B)$, and denoted by $M_{H(B)}$.

The study of the spectrum is simplified by fibering it over $\bar{B}^{**}$ (the closed unit ball of $X^{**}$) via the surjective mapping $\pi : M_{H(B)} \rightarrow \bar{B}^{**}$ given by $\pi(\tau) = \tau|_{X^*}$.
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Given $f \in H(B)$, the Gelfand Transform of $f$ is the continuous map

$$\hat{f} : M_{H(B)} \rightarrow \mathbb{C}$$

given by $\tau \mapsto \tau(f)$.

**Note:** The Gelfand Transform is a generalization of the Fourier Transform for $L_1(\mathbb{R})$ under convolution.
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Big open problems in the study of algebras $H(B)$:

**Corona problem:** Is $B$ dense in $M_{H(B)}$ (in the $w^*$ topology)?

**Remark**

$B \subset M_{H(B)}$ via $\delta : B \to M_{H(B)}$ such that $x \mapsto \delta_x$, where $\delta_x : H(B) \to \mathbb{C}$ is defined by $f \mapsto f(x)$. 

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Results related to Corona problem:

- Carleson, ’62: Corona theorem for the unit disk $\Delta \subset \mathbb{C}$.
- Gamelin, ’70; Garnett and Jones, ’85: Corona theorems for other planar domains.
- Sibony, ’87: Pseudoconvex counterexample in $\mathbb{C}^3$ to Corona problem.
- Sibony, ’93: Pseudoconvex and strictly pseudoconvex (except at one point) counterexample in $\mathbb{C}^2$ to Corona problem.

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Another set of big open problems in the study of algebras $H(B)$: Cluster value problems.

The *cluster value theorem for $H(B)$* asserts that, for a given $x^{**} \in \overline{B}^{**}$, the sets of cluster values

$$Cl_B(f, x^{**}) := \{ \lambda : f(x_\alpha) \to \lambda, \; x_\alpha \overset{w^*}{\to} x^{**} \}$$

coincides with the Gelfand transform of $f$ evaluated on the fiber $M_{x^{**}}(B) := \pi^{-1}(x^{**})$,

$$\hat{f}(M_{x^{**}}(B)) = \{ \tau(f) : \tau \in M_{x^{**}} \},$$

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The proof of this last result uses a solution to a $\bar{\partial}$ problem in strongly pseudoconvex domains (Kerzman, ’71).
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What happens when $B$ is the unit ball of an infinite-dimensional Banach space? Let us first overview the basic theory of analytic functions on arbitrary Banach spaces.

Given $U$ an open subset of a Banach space $X$, $f : U \to \mathbb{C}$ is analytic if for every $x \in U$ there exists $r > 0$ and continuous polynomials on $X$, $(P^m f(x))_{m=0}^{\infty}$, where $P^m f(x)$ is $m$-homogeneous, such that, if $\|y - x\| < r$ then $f(y) = \sum_{m=0}^{\infty} P^m f(x)(y - x)$, and the convergence is uniform on $B(x, r)$ (and $r_{cf}(x)$ is the supremum of such $r$).

An $m$-homogeneous polynomial $\hat{L}$ on $X$, for $m \in \mathbb{N}$, is the restriction to the diagonal of a $m$-linear mapping $L : X^m \to \mathbb{C}$, i.e. $\hat{L}(x) = L(x, \cdots, x)$ (and it is a constant function for $m = 0$).
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**Theorem**

Given $f : U \subset X \rightarrow Y$, TFAE:

- $f$ is analytic,
- $f$ is continuous and analytic on each complex line, i.e.

  $$
  \lambda \mapsto f(a + \lambda b) \text{ is analytic for all } a \in U \text{ and } b \neq 0 \in E,
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  on $\{\zeta \in \mathbb{C} : a + \zeta b \in U\}$,
- $f$ is Fréchet $\mathbb{C}$-differentiable.

**Example**

If $x_m^* \in X^*$ and $x_m^* \xrightarrow{w^*} 0$, then $\sum_{m=0}^{\infty} (x_m^*)^m$ is analytic on $X$. 

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Introduction: Main Concepts

Properties of analytic functions that extend to $\infty$-dimensional setting:
Open Mapping Principle, Maximum Principle, Liouville’s Theorem.

More properties of analytic functions that extend:
Given $f : U \subset X \rightarrow Y$ analytic, $a \in U$, $b \in X$, and $r$ small enough,
- $P^m f(a)(b) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(a + \zeta b)}{\zeta^{m+1}} d\zeta$ if $\lambda \in r\Delta$ and $m \in \mathbb{N}_0$,
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When $B$ is the unit ball of an infinite-dimensional Banach space, there are no positive solutions to the Corona problem.

Aron, Carando, Gamelin, Lasalle, Maestre, ’12 [1]:

- Cluster value theorem for $x = 0$ and $A_u(B)$, when $X$ has a shrinking $1$-unconditional basis.

**Example**

The previous condition is satisfied by $\ell_p$ for $1 < p < \infty$ and $c_0$, but not by $\ell_1, \ell_\infty$ nor $L_p(0, 1)$ for $1 \leq p \neq 2 \leq \infty$. 
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Remark

Hilbert space and $c_0$ are infinite-dimensional analogues of the unit ball and the polydisk of Euclidean space.

More known cluster value theorems:

Farmer, ’98: There is a cluster value theorem for each point in $\partial B$ and $A_u(B)$, when $B$ is the unit ball of a uniformly convex Banach space, like $\ell_p$ and $L_p$ for $1 < p < \infty$; Acosta and Lourenzo, ’07: There is a cluster value theorem for $A_u(B_{\ell_1})$ at each point in $\partial B_{\ell_1}$. 
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Generalizing the ideas and techniques of Aron, Carando, Gamelin, Lasalle and Maestre, we proved:

**Theorem (Johnson, O., ’13)**

Suppose that for each finite-dimensional subspace $E$ of $X^*$ and $\epsilon > 0$ there exists a finite rank operator $S$ on $X$ so that $\|(I - S^*)|_E\| < \epsilon$ and $\|I - S\| = 1$. Then the cluster value theorem holds for $A_u(B)$ at 0.

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If $X$ is a Banach space with a shrinking reverse monotone Finite Dimensional Decomposition, we have a cluster value theorem for $A_u(B)$ at 0.
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Our Cluster Value Theorems

We also proved the following relationship with the help of Aron and Maestre:

**Lemma (Aron, Maestre)**

If $Y$ is a closed finite-codimensional subspace of $X$ and $f \in A_u(B)$, then $Cl_B(f, 0) = Cl_{B_Y}(f|_Y, 0)$, where $B_Y$ is the unit ball of $Y$.

Since $c$ is one-codimensional in $c_0$ and $c_0$ satisfies a cluster value theorem, the previous result suggests that $c$ satisfies a cluster value theorem for $A_u(B)$.

It turns out that $A_u(B_c) = A(B_c)$ because $c$ is isomorphic to $c_0$, so $c$ indeed satisfies a cluster value theorem for $A_u(B_c)$.

Moreover, $A_u(B) = A(B)$ when $X = C(K)$ for any $K$ compact, Hausdorff and dispersed, so $X$ satisfies a cluster value theorem for $A_u(B)$. 
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It turns out that $A_u(B_c) = A(B_c)$ because $c$ is isomorphic to $c_0$, so $c$ indeed satisfies a cluster value theorem for $A_u(B_c)$.

Moreover, $A_u(B) = A(B)$ when $X = C(K)$ for any $K$ compact, Hausdorff and dispersed, so $X$ satisfies a cluster value theorem for $A_u(B)$. 
Our Cluster Value Theorems

We also proved the following relationship with the help of Aron and Maestre:

**Lemma (Aron, Maestre)**

If $Y$ is a closed finite-codimensional subspace of $X$ and $f \in A_u(B)$, then $\text{Cl}_B(f, 0) = \text{Cl}_{B_Y}(f|_Y, 0)$, where $B_Y$ is the unit ball of $Y$.

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Following the proof by Aron, Carando, Gamelin, Lasalle and Maestre that $c_0$ satisfies a cluster value theorem for $H^\infty(B)$, and using that $C(K)^* = \ell_1(K)$ when $K$ is compact, Hausdorff and dispersed, we obtain a cluster value theorem for $H^\infty(B)$ when $X = C(K)$, and $K$ is compact, Hausdorff and dispersed.

We consider the following in [2]: Given $f_0^{**} \in \overline{B}^{**}$, the cluster value problem for $H^\infty(B)$ over $A_u(B)$ at $f_0^{**}$ asks whether for all $\psi \in H^\infty(B)$ and $\tau \in \mathcal{M}_{f_0^{**}}(B)$, can we find a net $(f_\alpha) \subset B$ such that $\psi(f_\alpha) \to \tau(\psi)$ and $f_\alpha$ converges to $f_0^{**}$ in the polynomial-star topology (that we denote by $\tau(\psi) \in \text{Cl}_B(\psi, f_0^{**})$)?

**Theorem**

The cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at 0 is equivalent to the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at any $f_0 \in B$, for $X = C(K)$. 
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In [3] we prove that for any separable Banach space $Y$, a cluster value problem for $H(B_Y)$ ($H = H^\infty$ or $H = A_u$) can be reduced to a cluster value problem for $H(B_X)$ for some Banach space $X$ that is an $\ell_1$-sum of a sequence of finite-dimensional spaces.

In particular, if $H(B_{\ell_1})$ satisfies the cluster value theorem, then so does $H(B_{L_1})$.  

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Corona and cluster value p. in $\infty$-dim. spaces  
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In particular, if $H(B_{\ell_1})$ satisfies the cluster value theorem, then so does $H(B_{L_1})$. 
Does the cluster value theorem hold for $H^\infty(B)$ or $A_u(B)$, when $B = B_{\ell_1}$ or $B = B_X$ and $X$ is uniformly convex?

Remark
Lempert, '99: There is a solution to the $\overline{\partial}$ problem in $B_{\ell_1}$.

Is the previous solution weakly continuous?

Is there a solution to the $\overline{\partial}$ problem in $B_X$ for $X$ uniformly convex?
If so, is it weakly continuous?

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