

Are you sure that's an ellipse?
Poncelet ellipses, Blaschke products, and other
mathematical short stories.

Pamela Gorkin

Bucknell University

April 2014

From the papers:

- Three Problems in Search of a Measure, J. King, 1994 (Monthly)
- Poncelet's Theorem, The Sendov Conjecture, and Blaschke products (with U. Daepf and K. Voss, JMAA 2010)
- Numerical ranges of restricted shifts and unitary dilations (G., I. Chalendar and J. R. Partington, Operators and Matrices, 2010)
- The group of invariants, (G., I. Chalendar and J. R. Partington)

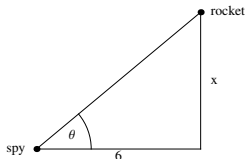
With an applet done with Keith Taylor and Duncan Gillis (Dalhousie) and an applet by Andrew Shaffer (Bucknell, Lewisburg)

What does it mean for two problems to “be the same”?

Problem 1.[Rogawski] A spy uses a telescope to track a rocket launched vertically from a launching pad 6 km away. At a certain moment the angle θ between the telescope and the ground is equal to $\pi/3$ and is changing at a rate of 0.9 rad/min. What is the rocket's velocity at that moment?

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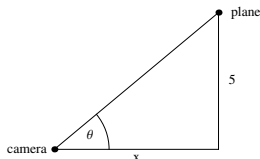
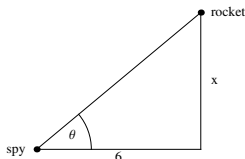
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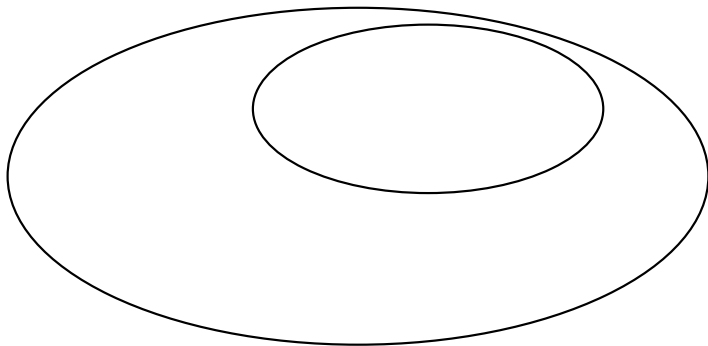
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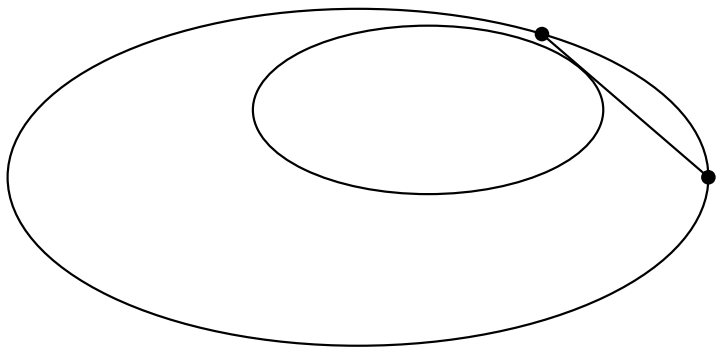


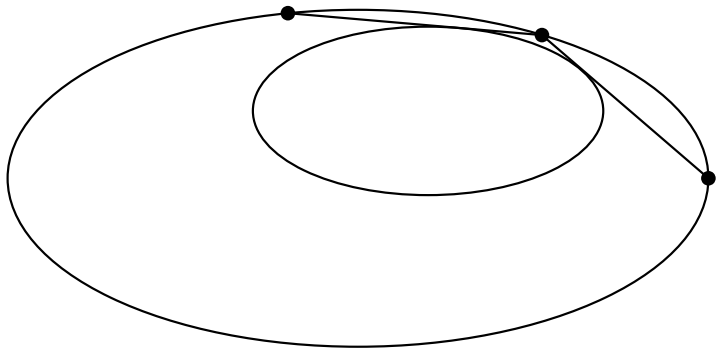
Problem 3.[Rogawski] The minute hand of a clock is 8 cm long and the hour hand is 5 cm long. How fast is the distance between the tips of the hands changing at 3 o'clock.

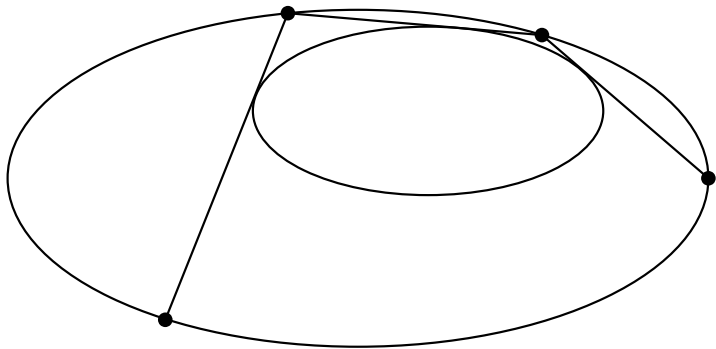
Poncelet's Theorem

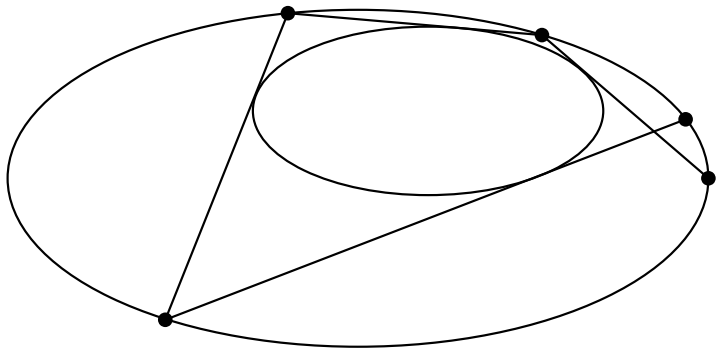
Given an ellipse, and a smaller ellipse entirely inside it, start at a point on the outer ellipse, and, follow a line that is tangent to the inner ellipse until you hit the outer ellipse again.

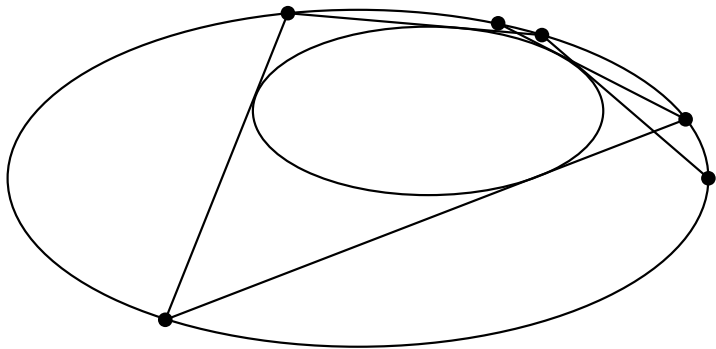


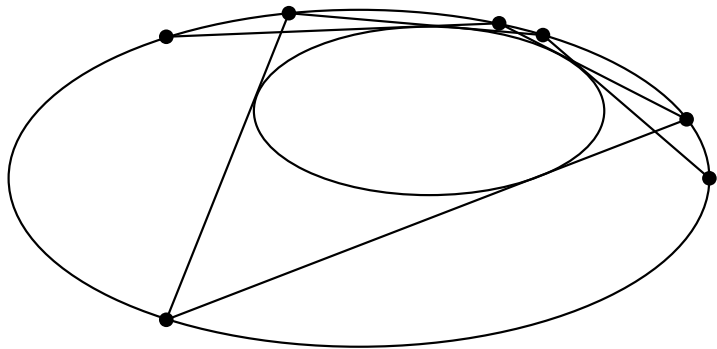


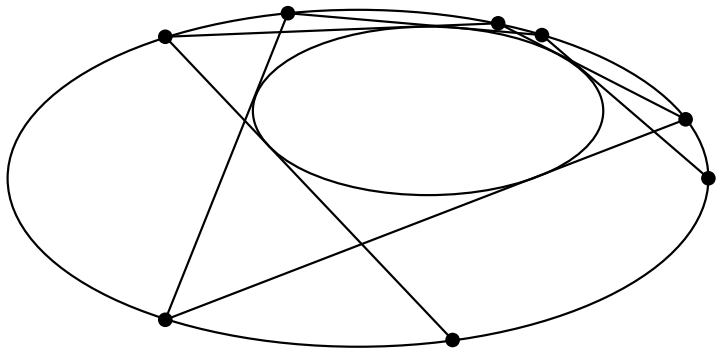


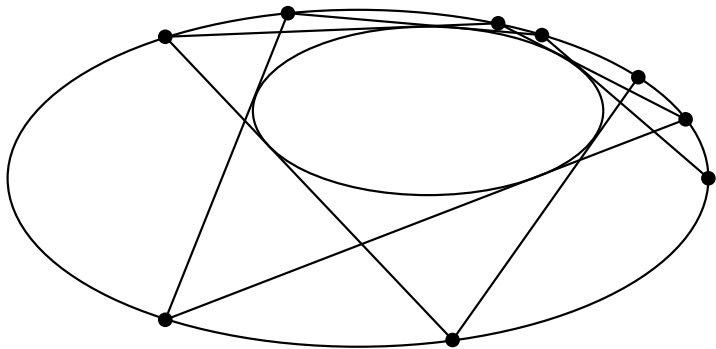




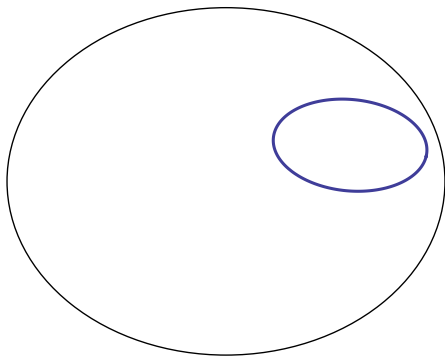


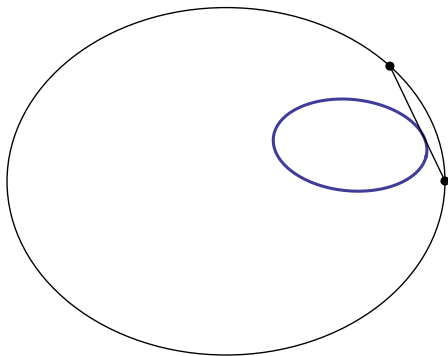


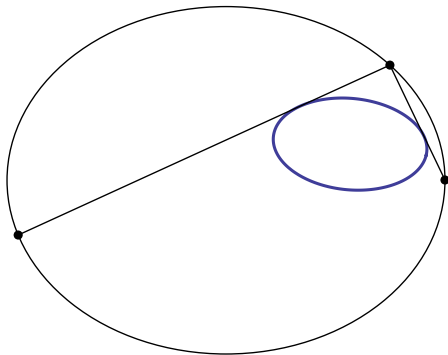


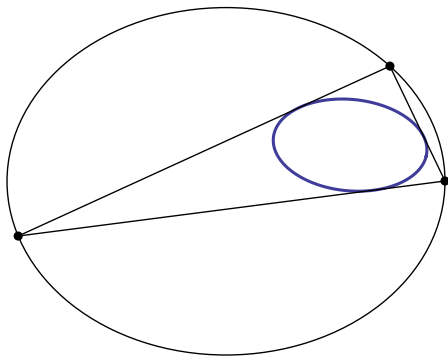


Maybe, though, it does close in n steps.



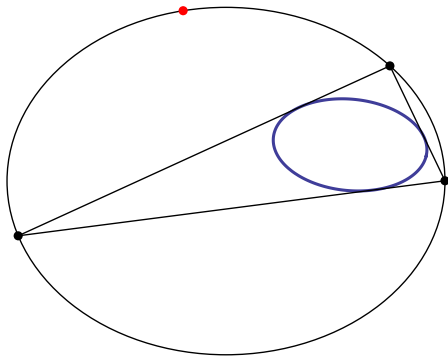


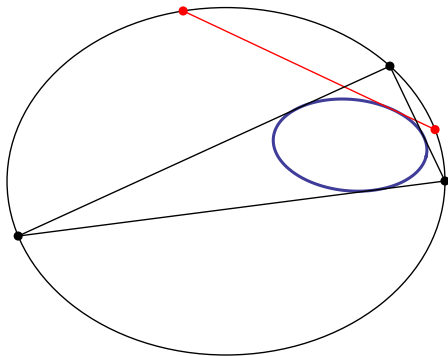


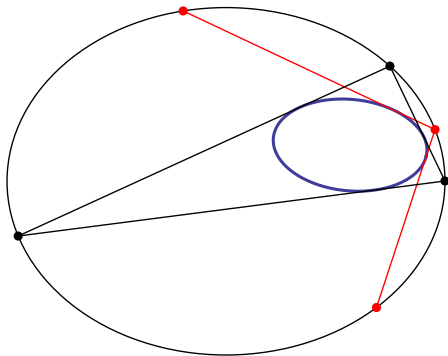


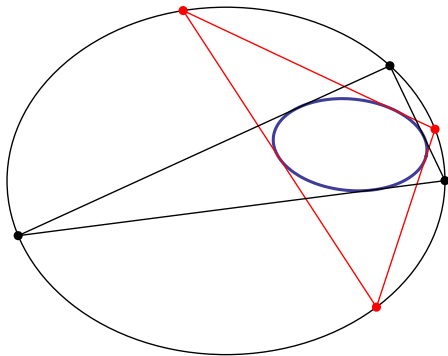
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This fact is Poncelet's theorem, also known as Poncelet's closure theorem, and is named after Jean Poncelet.

But what does this have to do with the real world?

After all, pool tables are not elliptical...

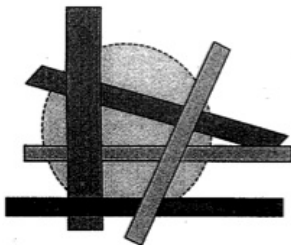
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Tarski's Plank Problem

Given a circular table of diameter 9 feet, which is the minimal number of planks (each 1 foot wide and length greater than 9 feet) needed in order to completely cover the tabletop? Nine parallel planks suffice, but is there a covering using fewer planks if suitably oriented?



More precisely...

Suppose (w_n) are widths of a countable family of planks covering \mathbb{D} . Then

$$\sum_{n=1}^{\infty} w_n \geq \text{Width}(\mathbb{D}).$$

If $\sum_{n=1}^{\infty} w_n = \text{Width}(\mathbb{D})$, the diameter of the disk, then the planks must actually be parallel.

Gelfand's questions

Row n has the leftmost digit of $2^n, 3^n, \dots$ when written in base 10.

TABLE P.3 The leftmost digits of powers.

n :	2^n	3^n	4^n	5^n	6^n	7^n	8^n	9^n
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Questions: Will 23456789 occur a second time? 248136?
(infinitely often in column 1, but in no other column).

Who cares about patterns of sequences in the real world?

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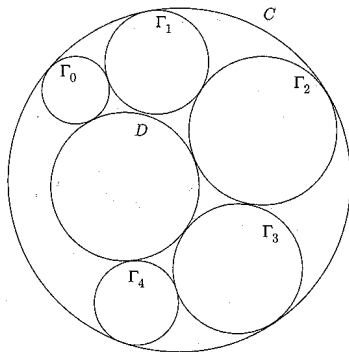
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Law applies to budget, income tax or population figures as well as street addresses of people listed in the book *American Men of Science*.

Wait...isn't this four problems?

Theorem (Steiner's Theorem)

Let C, D be circles, D inside C . Draw a circle, Γ_0 tangent to C and D . Then draw a circle tangent to C, D , and Γ_0 . Repeat, getting $\Gamma_0, \dots, \Gamma_n$. If $\Gamma_n = \Gamma_0$, then $\Gamma_n = \Gamma_0$ for all initial choices of Γ_0 .



Three of these are looking for a measure. Which three?

Poncelet's Theorem

If an n -sided Poncelet transverse constructed for two given conic sections is closed for one point of origin, it is closed for any position of the point of origin.

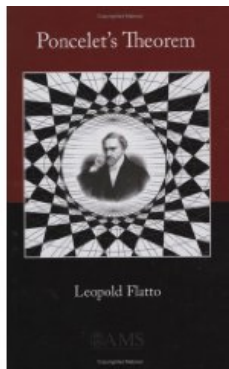
Specifically, given one ellipse inside another, if there exists one circumscribed (simultaneously inscribed in the outer and circumscribed on the inner) n -gon, then any point on the boundary of the outer ellipse is the vertex of some circumscribed n -gon.

Good references, easy reading

Poncelet's Theorem, Leopold Flatto 2009

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What's a porism?

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The term "porism" is an archaic type of mathematical proposition whose historical purpose is not entirely known. It is used instead of "theorem" by some authors for a small number of results for historical reasons. .—J. K. Barnett, Wolfram

Easy case: The unit disk and the circle $\{z : |z| = 1/2\}$.

Easy because we simply look for arcs of equal length.

Idea: Find a measure that assigns equal length to the arcs you get.

More precisely

Start at one point, z_0 , and draw a tangent. Let $R(z_0) = z_1$.

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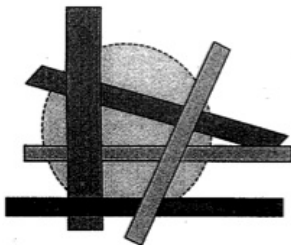
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We'll “solve” this problem for $n = 3$ and show the measure later.

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we can find a measure that is invariant with respect to rigid motions...

Getting our motivation from Poncelet

We need an area invariant measure and we expect to integrate against a measure.

So, subsets of zero ν -measure should be of zero area and ν should be invariant under rigid motions;
 ν should depend only on the width, so

$$\nu(P) = \alpha \text{ Width}(P).$$

Remark: A plank is $P \cap \mathbb{D}$, so \mathbb{D} is a plank.

Partial “Proof” of the Plank Conjecture.

Since \mathbb{D} is a Plank,

$$\text{Width}(\mathbb{D}) = (1/\alpha)\nu(\mathbb{D}) = (1/\alpha)\nu(\cup_n P_n).$$

So

$$\text{Width}(\mathbb{D}) \leq (1/\alpha) \sum_n \nu(P_n) = \sum_n \text{Width}(P_n).$$

Thus, no cover can use less than the total width of \mathbb{D} .

It remains to show that only parallel covers use minimum width.

Involves showing that a radial projection is area-preserving. (Bang, 1950/1)

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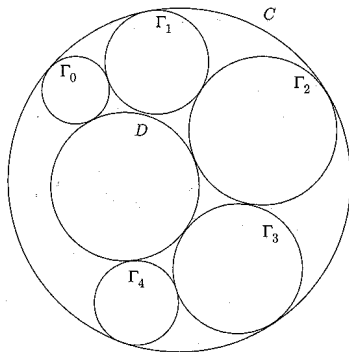
And this is, more or less, the picture we saw in Poncelet's theorem.

King's paper describes these problems as “problems in search of a measure.”

This one is not

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JOHANNES KEPLER'S UPHILL BATTLE



Flatto, Poncelet's Theorem, 2009

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Poncelet's theorem cannot.

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Right now.

Degree n -Blaschke products

$$B(z) = \prod_{j=1}^n \frac{z - a_j}{1 - \overline{a_j}z},$$
$$a_1, \dots, a_n \in \mathbb{D}$$

What is the group of invariants of a Blaschke product?

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Question: What is the group of continuous functions $u : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$ such that

$$B \circ u = B?$$

Note: If $B(z_1) = \lambda$, then $B(u(z_1)) = B(z_1) = \lambda$, so u permutes points in $B^{-1}\{\lambda\}$.

Degree-three Blaschke products

Consider

$$B(z) = z \left(\frac{z - a_1}{1 - \overline{a_1}z} \right) \left(\frac{z - a_2}{1 - \overline{a_2}z} \right).$$

- B maps the unit circle in the three-to-one fashion onto itself;

Blaschke products revealed

Given $\lambda \in \partial\mathbb{D}$, what can we say about the points where $B = \lambda$?

Taking the logarithmic derivative (derivative of $\log(B(z))$):

$$\begin{aligned} z \frac{B'(z)}{B(z)} &= z \left(\frac{1}{z} + \frac{1}{z - a_1} + \frac{\overline{a_1}}{1 - \overline{a_1}z} + \frac{1}{z - a_2} + \frac{\overline{a_2}}{1 - \overline{a_2}z} \right). \\ &= 1 + \frac{1 - |a_1|^2}{|1 - \overline{a_1}z|^2} + \frac{1 - |a_2|^2}{|1 - \overline{a_2}z|^2}. \end{aligned}$$

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So a Blaschke product never reverses direction and the set $E_\lambda = \{z \in \partial\mathbb{D} : b(z) = \lambda\}$ consists of three distinct points.

Blaschke products revealed again

Blaschke Ellipses

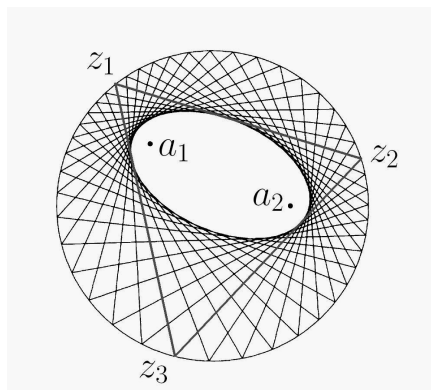


Figure : $b(z) = z \left(\frac{z-a_1}{1-\overline{a_1}z} \right) \left(\frac{z-a_2}{1-\overline{a_2}z} \right).$

Theorem (Daepp, G, Mortini, 2002)

Consider a Blaschke product b with zeros $0, a_1, a_2 \in \mathbb{D}$. For $\lambda \in \partial\mathbb{D}$, let z_1, z_2, z_3 denote the distinct points mapped to λ under b . Then the line joining z_1 and z_2 is tangent to

$$E : |w - a_1| + |w - a_2| = |1 - \overline{a_1}a_2|.$$

Conversely, each point of E is the point of tangency with E of a line that passes through points z_1 and z_2 on the circle for which $b(z_1) = b(z_2)$.

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Note: These ellipses are Poncelet ellipses.

What are the Poncelet 3-ellipses?

Theorem (Daepp, G, Mortini, 2002)

Consider a Blaschke product b with zeros $0, a_1, a_2 \in \mathbb{D}$. For $\lambda \in \partial\mathbb{D}$, let z_1, z_2, z_3 denote the distinct points mapped to λ under b . Then the line joining z_1 and z_2 is tangent to

$$E : |w - a_1| + |w - a_2| = |1 - \overline{a_1}a_2|.$$

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What are the Poncelet 3-ellipses?

They are precisely the ones associated with Blaschke products (Frantz, Monthly 2005)

This reminds me of...

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Theorem (Bôcher, Grace $m_j > 0$)

Let

$$F(z) = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3},$$

where m_1, m_2, m_3 are positive numbers, and z_1, z_2, z_3 are distinct complex numbers.

Then the zeros a_1 and a_2 of F are the foci of the ellipse that touches the line segments z_1z_2 , z_2z_3 , z_3z_1 in the points $\zeta_1, \zeta_2, \zeta_3$ that divide these segments in ratios $m_1 : m_2$, $m_2 : m_3$ and $m_3 : m_1$, respectively.

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General m_j due to M. Marden.

What's the connection?

Let $\lambda \in \partial\mathbb{D}$ with $b(z_j) = \lambda$:

$$\frac{b(z)/z}{b(z) - \lambda} = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3}$$

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$$m_j = b(z_j)/(z_j b'(z_j)) = 1/(1 + \frac{1 - |a_1|^2}{|1 - \overline{a_1} z_j|^2} + \frac{1 - |a_2|^2}{|1 - \overline{a_2} z_j|^2})$$

By the previous theorem: the zeros of $b(z)/z$ are the foci of the ellipse that touches the line segments $z_1 z_2$, $z_2 z_3$, $z_3 z_1$ in the points $\zeta_1, \zeta_2, \zeta_3$ that divide these segments in ratios $m_1 : m_2$, $m_2 : m_3$ and $m_3 : m_1$, respectively.

Poncelet's theorem

Given an ellipse inside the unit circle, how do we enclose it in a triangle?

We look for points that have equal “length” with respect to the measure

$$h(z) = z \frac{B'(z)}{B(z)} = 1 + \frac{1 - |a_1|^2}{|z - a_1|^2} + \frac{1 - |a_2|^2}{|z - a_2|^2}.$$

We've answered King's measure problem for 3-inscribed ellipses.

Back to the group of automorphisms

So, can we also answer our question for $n = 3$? That is, what is the group of continuous $u : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$ with $B \circ u = B$ for degree 3 Blaschke products?

Back to the group of automorphisms

So, can we also answer our question for $n = 3$? That is, what is the group of continuous $u : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$ with $B \circ u = B$ for degree 3 Blaschke products?

It's the cyclic group of order 3 and you've been looking at it in the applet.

What about degree n Blaschke products of the form

$$b(z) = z \prod_{j=1}^{n-1} \frac{z - a_j}{1 - \overline{a_j}z}?$$

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What about infinite Blaschke products?

i.e. Zeros (a_n) in \mathbb{D} ,

$$\sum_n (1 - |a_n|) < \infty$$

Commercial Break: A definitely different problem

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Theorem (Siebeck)

Suppose that the vertices of a triangle are z_1, z_2 , and z_3 . Let $p(z) = (z - z_1)(z - z_2)(z - z_3)$. Then the roots of p' are the foci of the inellipse of $\triangle z_1 z_2 z_3$, tangent to the sides at the midpoints.

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The zeros of the derivative of a polynomial are contained in the convex hull of the zeros of the polynomial.

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Sendov conjecture: Given a polynomial p with zeros inside the closed unit disk, for each zero z_0 of the polynomial is there a zero of the derivative within the circle $|z - z_0| \leq 1$?

The definitely different problem and its connection to Blaschke products

Let

$$p'(z) = 3(z - a_1)(z - a_2)$$

and a_1, a_2 lie in the open unit disk; zeros of p on the unit circle.

The definitely different problem and its connection to Blaschke products

Let

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$$\frac{p'(z)}{3p(z)} = (1/3) \sum_{j=1}^3 \frac{1}{z - z_j}.$$

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Bôcher Grace said: Let

$$F(z) = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3},$$

m_1, m_2, m_3 positive, and z_1, z_2, z_3 are distinct complex numbers.

But how do we get to a Blaschke product?

The connection: $p'(z) = 3(z - a_1)(z - a_2)$ and a_1, a_2 lie in the open unit disk; zeros of p on the unit circle.

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There is a Blaschke product b such that

$$\frac{b(z)/z}{b(z) - \lambda} = \frac{p'(z)}{3p(z)} = (1/3) \sum_{j=1}^3 \frac{1}{z - z_j}.$$

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Zeros of p' are the foci of an ellipse inscribed in the triangle $\Delta_{z_1 z_2 z_3}$.

Where are we on Sendov's conjecture?

- ① Degrees 3, 4 and 5 were solved relatively quickly; degree 6 was solved in 1996 (J. Borcea) and later degree 7.
- ② Degree 8 (J. Brown and G. Xiang, 1999)
- ③ All zeros real (Rahman, Schmeisser)
- ④ All zeros on the unit circle (Goodman, Rahman, Ratti; Schmeisser, 1969)
- ⑤ One zero at the origin (Bojanov, Rahman, Szynal, 1985)
- ⑥ In a different direction, moving zeros just a bit (M. Miller); recent quantitative estimates for this.

Borcea's variance conjectures on the critical points of polynomials

Khavinson, Pereira, Putinar, Saff and Shimorin dedicated to J. Borcea

And more recent results

In 2014, Jérôme Dégot [PAMS] proved a Sendov Conjecture for high degree polynomials

Theorem

Let P be a polynomial with zeros in the closed unit disk and suppose $P(a) = 0$, where $0 < a < 1$. Then there exists a constant N such that if the degree of P is bigger than N , then the derivative of P has a zero in the disk of radius 1 about a .

What is N ? It is defined in terms of three other constants: N_1 , N_2 and K .

And you should know that...

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N_1 is the smallest integer such that

$$\left(\frac{1+a/2}{1+a}\right)^q \leq \left(\frac{1-\sqrt{1-a^2/4}}{na}\right)^{1/(n-1)} \quad \text{for all } n \geq N_1,$$

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N_2 is the smallest integer such that

$$D^{n-1} \leq \frac{a}{16n} \quad \text{for all } n \geq N_2,$$

where

$$D := \max\left\{\left(\frac{1}{1+a}\right)^q : \left(\frac{1+c}{1+a}\right)^q (\sqrt{1+c^2-ac})^{1-q}\right\} < 1,$$

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and K doesn't get any better.

“It may be surprising to see that Sendov’s conjecture is easily proved in extremal cases, meaning when $a = 0$ or $a = 1$ and in the generic case, $0 < a < 1$, only a few partial results are known. In the present paper, we try to fill this gap, but it remains to obtain a definitive proof of the conjecture, that is, to demonstrate that, with our notations, $N = 8$ for all $a \in (0, 1)$.

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We have shown that if a zero, denoted by a , of P is given, one can compute an integer bound N , such that if $\deg P \geq N$, then P' has a zero in the disk $|z - a| \leq 1$. It would be nice if N could be given independently of $|a|$ or, at least, to have an explicit formula for N in terms of a .”

Back to our problem: What happens for higher degrees?

Blaschke products revealed again

Can we generalize this?

Is there a generalization of the Bôcher-Grace?

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Is there a generalization of the Bôcher-Grace?

Theorem (Siebeck, 1864)

The zeros of the function

$$F(z) = \sum_{j=1}^n \frac{m_j}{z - z_j},$$

where m_j real, are the foci of the curve that touches each line-segment $z_j z_k$ in a point dividing the line segment in the ratio $m_j : m_k$.

So...maybe. But the curve will be more complicated.

The matrix connection

Given an $n \times n$ matrix A , the numerical range of A is

$$W(A) = \{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}.$$

What happens if A has eigenvalue λ ?

Let's let x be a unit eigenvector.

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Theorem (C. K. Li, noncomputational proof)

*Given a 2×2 matrix with eigenvalues a_1 and a_2 , the numerical range of A is an elliptical disk with foci a_1 and a_2 and minor axis $\sqrt{\operatorname{tr}(A^*A) - |a_1|^2 - |a_2|^2}$.*

Basic facts about the numerical range of $n \times n$ matrices:

Basic facts about the numerical range of $n \times n$ matrices:

- ① $W(A)$ is convex (Toeplitz-Hausdorff theorem, 1918/1919), compact.
- ② $W(A)$ contains the spectrum of A . (M. Stone, 1932)
- ③ If A is normal ($A^*A = AA^*$), the extreme points of $W(A)$ are the eigenvalues (1957)
- ④ If A is normal, $W(A)$ is the closed convex hull of its eigenvalues.

Gau and Wu, 1998, 1999, 2000, 2003, 2004

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Consider the matrix A , eigenvalues $a_1, a_2 \in \mathbb{D}$

$$\begin{pmatrix} a_1 & \sqrt{1 - |a_1|^2} \sqrt{1 - |a_2|^2} \\ 0 & a_2 \end{pmatrix}$$

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- Then A is a contraction ($\|A\| \leq 1$);
- eigenvalues are a_1, a_2 —the zeros of the Blaschke product we considered
- A dilates to a unitary operator ($U^*U = UU^* = I$) on a space K :
unitary: columns form an orthonormal basis for K and
special property of our matrix:

$$\dim(K \ominus H) = \operatorname{rank}(1 - A^*A) = 1.$$

$$B_\lambda = \begin{bmatrix} a_1 & \sqrt{1-|a_1|^2}\sqrt{1-|a_2|^2} & -\overline{a_2}\sqrt{1-|a_1|^2} \\ 0 & a_2 & \sqrt{1-|a_2|^2} \\ \lambda\sqrt{1-|a_1|^2} & -\lambda\overline{a_1}\sqrt{1-|a_2|^2} & \lambda\overline{a_1}a_2 \end{bmatrix}$$

where $|\lambda| = 1$.

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where $|\lambda| = 1$.

B_λ a unitary dilation of A : $V^*B_\lambda V = A$,

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The characteristic polynomials of A and B_λ

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Of A : $q(z) = (z - a_1)(z - a_2)$;

Of B_λ : $p(z) = z(z - a_1)(z - a_2) - \lambda(1 - \overline{a_1}z)(1 - \overline{a_2}z)$.

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i. e. eigenvalues are where

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$$\frac{z(z - a_1)(z - a_2)}{(1 - \overline{a_1}z)(1 - \overline{a_2}z)} = \lambda.$$

So: The eigenvalues of B_λ are the three (distinct) values b maps to λ .

Note: Every 3×3 unitary matrix with distinct eigenvalues is unitarily equivalent to a B_λ .

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- $W(A)$ is an *elliptical disk* with *foci* at the eigenvalues of A or the zeros of $B(z)/z$.

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- $W(B_\lambda)$ is the convex hull of the eigenvalues of B_λ or *the triangle formed by the three distinct points identified by the Blaschke product*.

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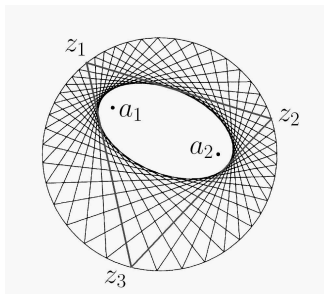


Figure : $W(A) = \bigcap_{\lambda \in \mathbb{D}} W(B_\lambda)$.

Where we are

- ① Poncelet's theorem for 3-gons (triangles);
- ② Connected points identified by degree-3 Blaschke products;
- ③ Connection to a theorem of Bôcher-Grace and the Sendov conjecture;
- ④ Numerical range of a 2×2 matrix and its 3×3 dilations.

The matrices A and B_λ

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$$A = \begin{pmatrix} a_1 & \sqrt{1-|a_1|^2}\sqrt{1-|a_2|^2} & \dots & (\prod_{k=2}^{n-1}(-\overline{a_k}))\sqrt{1-|a_1|^2}\sqrt{1-|a_n|^2} \\ 0 & a_2 & \dots & (\prod_{k=3}^{n-1}(-\overline{a_k}))\sqrt{1-|a_2|^2}\sqrt{1-|a_n|^2} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & a_n \end{pmatrix}$$

$$B_\lambda = \begin{pmatrix} & & & \prod_{k=2}^n(-\overline{a_k})\sqrt{1-|a_1|^2} \\ & & & \prod_{k=3}^n(-\overline{a_k})\sqrt{1-|a_2|^2} \\ & & & \dots \\ & A & & \\ \lambda\sqrt{1-|a_1|^2} & \dots & \lambda(\prod_{k=1}^{j-1}(-\overline{a_k}))\sqrt{1-|a_j|^2} & \dots & \lambda\prod_{k=1}^n(-\overline{a_k}) \end{pmatrix}$$

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Example.

$$\begin{pmatrix} 0 & & & & \\ 1 & \cdot & & & \\ & \dots & \dots & & \\ & & \cdot & \cdot & \\ 0 & & & 1 & 0 \end{pmatrix}$$

Jordan block size n ; numerical range $\{z : |z| \leq \cos(\pi/(n+1))\}$
(Haagerup, de la Harpe, 1992)

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As before, B_λ is a unitary dilation of A (for each λ)

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Theorem (Gau, Wu, 1998)

If $T \in S_n$, then for $\lambda \in \partial\mathbb{D}$ there is a unique n -gon P such that:

- ① P is inscribed in $\partial\mathbb{D}$;*
- ② P is circumscribed about $W(T)$ with each side tangent at precisely one point;*
- ③ P has λ as a vertex.*

This is, again, a version of Poncelet's theorem for n -gons.

The group of invariants for a degree- n Blaschke product

Theorem (Cassier, Chalendar)

Let B be a finite Blaschke product. Then the group of invariants is \mathbb{Z}_n , the cyclic group of order n .

Infinite products, with I. Chalendar and J. R. Partington

$$b(z) = z^m \prod \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z}, \quad \sum (1 - |a_n|) < \infty.$$

Infinite products, with I. Chalendar and J. R. Partington

$$b(z) = z^m \prod \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z}, \quad \sum (1 - |a_n|) < \infty.$$

b is bounded and analytic on \mathbb{D} , maps \mathbb{D} to \mathbb{D} , but does not have modulus one on the unit circle.

“ $|b| = 1$ ” almost everywhere on the unit circle, though, but this is a roadblock for our problem.

There's also a second “type” of function that “acts like” an infinite Blaschke product and those are called singular inner functions. They are part of the class of **inner functions**.

Other inner functions

A **singular inner function** is a bounded analytic function with no zeros in the disk, maps the open unit disk to itself, and has radial limits of modulus one almost everywhere on the unit circle.

Bounded analytic functions that have modulus one a.e. on the unit circle are called **inner functions** and every inner function I is

$$I = BS,$$

where B is Blaschke and S is singular inner.

$$S(z) = \exp \left(- \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right)$$

where the measure is singular with respect to Lebesgue measure.

So is there an infinite version of Poncelet's theorem?

Yes. But...we need to think about what we can ask.

So let's suppose there's only one singularity; i.e. one bad point that the zeros approach.

And here's “proof” that it will work!

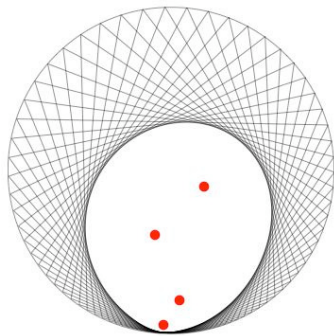
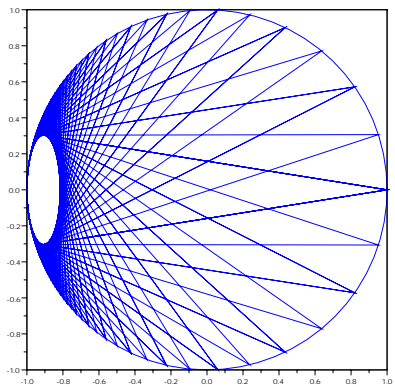


Figure : Blaschke product with one singularity



Recommended Reading

- ① Bercovici, Hari; Timotin, Dan, The numerical range of a contraction with finite defect numbers,
<http://arxiv.org/pdf/1205.2025.pdf>.
- ② Courtney, Dennis; Sarason, Donald, *A mini-max problem for self-adjoint Toeplitz matrices*, Math. Scand. 110 (2012), no. 1, 82–98.
- ③ Gorkin, P., Skubak, Beth, *Polynomials, ellipses, and matrices: two questions, one answer*, Amer. Math. Monthly 118 (2011), no. 6, 522–533.

Thank you!