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INFORMAL ANALYSIS SEMINAR/Lecture Series in Analysis
Saturday and Sunday, February 24-28, 2018

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LECTURES: Mathematical Sciences Building, Room 228.
POSTER SESSION/REFRESHMENTS/LUNCHES: SCALE-UP Mathematics Lab, room 319 Mathematical Sciences Bldg, Third floor.

The Mathematical Sciences Building is located on Summit Street, Kent, OH 44242. To search it on Goggle Maps, use the address 1400 East Summit Street, Kent, Ohio 44240.

Saturday, February 24

11:00 - 11:30	Coffee in Room 319 MSB.
11:30 - 12:30	Robert Connelly
12:30 - 1:30	Lunch in Room 319 MSB
1:30 - 2:30	Peter Sternberg
2:30 - 3:30	Break/Poster Session
3:30 - 4:30	Robert Connelly
4:30 - 5:00	Break
5:00 - 6:00	Peter Sternberg
6:30pm	Dinner: Wild Papaya Thai Cuisine (1665 E Main St, Kent, OH).

Sunday, February 25

9:00 - 9:30	Coffee in Room 319 MSB.
9:30 - 10:30	Peter Sternberg
10:30 - 10:45	Break
10:45 - 11:45	Robert Connelly
11:45 - 12:30	Lunch in Room 319 MSB
12:30 - 1:30	Peter Sternberg
1:30 - 1:45	Break
1:45 - 2:45	Robert Connelly

A degenerate isoperimetric problem in the plane with applications to a bi-stable Hamiltonian system.

Peter Sternberg

I will describe joint work with Stan Alama, Lia Bronsard, Andres Contreras and Jiri Dadok giving criteria for existence and for non-existence of certain isoperimetric planar curves minimizing length with respect to a metric having conformal factor that is degenerate at two points. More specifically, consider a smooth, non-negative map $W : \mathbb{R}^2 \to [0, \infty)$ vanishing at precisely two points, say $\mathbf{p}_-$ and $\mathbf{p}_+$. Then taking $F := \sqrt{W}$ as a conformal factor, we pursue the existence of a planar isoperimetric curve $\gamma : [a, b] \to \mathbb{R}^2$ that is required to join these two potential wells. Thus, the problem takes the form

$$\inf \left\{ \int_a^b F(\gamma(t)) \left|\gamma'(t)\right| \, dt : \ \gamma(a) = \mathbf{p}_-, \, \gamma(b) = \mathbf{p}_+, \, \int_\gamma \omega_0 = A \right\}, \tag{1}$$

where ω_0 is the 1-form $p_1 dp_2$, and so the interpretation as a Euclidean area constraint comes through an application of Stokes Theorem to the closed curve $\gamma \cup \gamma_0$ in that $d\omega_0 = dp_1 \, dp_2$. An equivalent version of the problem results from replacing ω_0 by any 1-form ω such that $d\omega = dp_1 \, dp_2$.

What makes this particular isoperimetric problem non-standard is both the degeneracy of the conformal factor F and the fact that length is measured with respect to a metric given by F while area is measured with respect to the Euclidean metric.

It turns out that this innocent looking problem carries some subtleties. For example, if W fails to satisfy a non-degeneracy condition at $\mathbf{p}_+$ and $\mathbf{p}_-$ in the form of a positive definite Hessian, then it is easy to construct counterexamples to existence. Minimizing sequences for (1) tend to wind more and more around the wells, converging weakly to a limiting curve that violates the area constraint. Perhaps more surprisingly, I will describe non-existence examples even *with* the non-degeneracy condition holding for W when the area constraint value A is too big.

The existence of these geodesics also carries some interesting surprises: in many cases the optimal geodesic solving (1) consists of an infinite (logarithimic) spiral. This is one of a variety of observations that will be revealed through a careful examination of the special case where W (and hence F) is assumed to be radial (i.e. $W = W(|p|)$ in neighborhoods of the two wells $\mathbf{p}_+$ and $\mathbf{p}_-$.

Using the radial results as a stepping stone, I will then present the existence result for general potentials W, that is for those that are not necessarily radial near the wells . The proof hinges on a recasting of the problem (1) as a variational problem posed for curves $\Gamma : [a, b] \to \mathbb{R}^3$, where the extra third dimension keeps track of the amount of Euclidean area accumulated by the projection of Γ into the plane as it traverses a path from $\mathbf{p}_-$ to $\mathbf{p}_+$.

Our primary motivation in pursuing this isoperimetric problem is that the geodesics solving (1), appropriately parametrized, emerge as traveling waves for the bi-stable Hamiltonian system:

$$\mathbb{J} u_t = \Delta u - \nabla_u W(u) \quad \text{for } u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^2 \tag{2}$$

where $\mathbb{J}$ denotes the symplectic matrix given by

$$\mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This not yet well-understood system of partial differential equations represents a form of conservative dynamics associated with the well-studied vector Allen-Cahn (or vector Modica-Mortola) energy

$$u \mapsto \int \frac{1}{2} |\nabla u|^2 + W(u),$$

and could provide an interesting model for conservative phase transitions in the Hamiltonian setting. One suspects that having a solid understanding of traveling wave solutions will be a helpful first step in subsequent analysis of the dynamics associated with (2). Invoking the traveling wave ansatz

$$u(x_1, \ldots, x_n, t) = U(x_1 - \nu t)$$

in (2), one sees that $U : \mathbb{R} \to \mathbb{R}^2$ must solve the system of ODE's

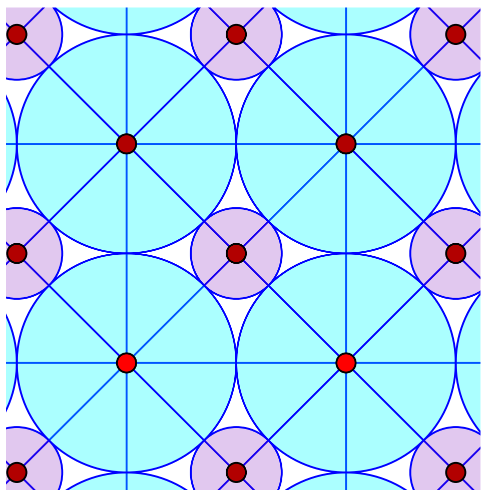
$$-\nu \mathbb{J} U' = U'' - \nabla_u W(U) \quad \text{for } -\infty < y < \infty, \quad U(\pm\infty) = \mathbf{p}_{\pm}. \tag{3}$$

It turns out that isoperimetric curves γ solving (1), appropriately parametrized, will solve this system where the wave speed ν depends on A and emerges as a Lagrange multiplier associated with the Euclidean area constraint. This connection between the two problems (1) and (3), along with some background on the traditional (i.e. dissipative) Allen-Cahn system, will be explained in the first lecture and further developed in the last one once existence of geodesics solving (1) has been established.

Packing a Torus with Circles.

Robert Connelly

Suppose you have a finite collection of circular disks of various sizes, and you want to fit them in a container without overlap so they occupy the largest fraction of the area possible. Think of the container as a torus, which is just the plane wrapped up by two independent translations. An example is the following, where there are two sizes of disks 1 and $\sqrt{2}-1$. Notice that the *graph of the packing* (obtained by joining centers of touching disks) is a triangulation in the plane. This property often seems to be evident for efficiently packed disks.



The packing fraction or *density* of this packing is $\pi(1-\sqrt{2}) = 0.92015\ldots$. In 2000 Aladar Heppes showed that this density is the largest possible using these two sizes of disks.

Problems of this sort have connections to rigidity theory (my day job), granular materials, approximate conformal (angle preserving) mappings given by packings, number theory, and salt, all explained. We extend a program of László Fejes Tóth, the great Hungarian geometer, with a conjecture about the most dense packings for some classes of circle packings in a torus.

Another aspect of these ideas has to do with jammed packings of disks in a fixed container, where the radii ratios of the disks are essentially chosen randomly, as you might expect with some granular materials. Given the number of disks, rigidity theory tells us that there has to be minimum number of contacts among the disks. It has been a long standing conjecture (the *isostatic conjecture*), and taken as obvious by many scientists, that generally you cannot have more than that minimum number of contacts. The analytic theory of packings as used by Koebe, Andreev, and Thurston gives us a way of actually proving that that the isostatic conjecture is correct.

A WARM-UP PUZZLER: Find a (doubly periodic) planar packing of circular disks, with a finite number of different sizes, whose graph is a triangulation with density a rational multiple of π.

Tensegrities: Why things don't fall down.

Robert Connelly

Suppose you have a finite collection of points in Euclidean space or the plane. Some pairs are connected by inextendible cables, others by incompressible struts, and some by fixed length bars. The artist Kenneth Snelson has constructed several large structures, made of cables and bars, that hold their shape under tension, where the struts appear to be suspended in midair. Buckminster Fuller, the architect and inventor, called them *tensegrities* because of their "tensional integrity". The Figure is one of Snelson's many tensegrity creations.

V-X, 1968
stainless steel
72 x 120 x 120 in
182.9 x 304.8 x 304.8 cm

But why do they hold their shape? There is a very simple principle using quadratic energy functions that provides the key to their stability. The tensegrity in the Figure is a typical application. We will show a catalog of highly symmetric tensegrities, created with the help of a little bit of representation theory, as well as tangible models, where you can feel their rigidity first-hand.