The critical probability for Voronoi percolation.

Thesis for the M.Sc. Degree

by

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Abstract

We consider a Voronoi percolation model on $I\!R^2$ (model was suggested by I. Benjamini and O. Schramm [2], [3]). This model should be presented as one of a continuum analogue of the discrete percolation model. We present two results. First we show uniqueness of infinite open cluster. Next we prove that if $p = \frac{1}{2}$, then the probability of having infinite open cluster is zero.

Contents

Chapter 1

Introduction

The word *percolation*, borrowed from the Latin, refers to the seeping or oozing of a liquid trough a porous medium, usually to be strained. In this and related senses it has been in use since the seventeenth century. Percolation was introduced into mathematics by S.R Broadbent and J.M. Hamnesley and is a branch of probability theory that is especially close to statistical mechanics. A percolation process typically depends on one or more probabilistic parameters.

Consider several standard models of percolation. The first model is referred to as the lattice model (bond percolation model), which means that is attached to sites on a square lattice. Let Z^2 be the plane square lattice and let p be a number satisfying $0 \leq p \leq 1$. We examine each edge of Z^2 in turn, and declare this edge to be open with probability p or closed with probability $1-p$, independently of all other edges. There are several basic questions that we may deal with in connection to the lattice model. One of them is related to the hypothesis of universality and conformal invariance (see Langlands, Pouliot, Saint-Aubin [13]). The other question is about critical value of parameter p. We define W to be an event that there exists an infinite open cluster starting from zero. Then, one should look for the critical value p_c for the edge density p such that all open clusters are finite when $p < p_c$, but there exists an infinite open cluster when $p > p_c$. It turns out that $p_c = \frac{1}{2}$ for the lattice percolation in Z^2. There are several ways to prove this fact. First way is by using Russo-Semour-Welsh theorem (in the sequel the RSW Theorem; see Russo [17] , Seymour and Welsh [18]). This theorem relates a probability of the left-right open crossing for a $L \times L$ square to that for a $(\frac{3L}{2}) \times L$ rectangle. Roughly, the RSW Theorem says that, for some function $f : (0,1] \to (0,1]$ with $f(q) \to 1$, as $q \to 1$, if the probability is at least q for crossing the square, then it is at least $f(q)$ for the left-right crossing of the rectangle, and the result is uniform in L. Kesten [11] generalized this theorem in a way that allows the square to be replaced by a $cL \times L$ rectangle, with $c < 1$, though $\frac{3}{2}$ should be replaced with a smaller constant $c' > 1$. Using the RSW theorem, one can prove that $p_c \geq \frac{1}{2}$. However, there is the other way to prove the same fact using the uniqueness property of the infinite open path. In the present work we are going to use the ideas of that proof.

The second standard model of percolation is usually referred to as the continuum percolation model (blob model). Instead of side/bonds being independently occupied or vacant we have a Poisson process on $\mathbb{R}^2$ with each Poisson point being the center of an 'occupied' disc of random radius. We assume that the random variables describing the radii of the discs are i.i.d., strictly positive and bounded above by a positive constant.

This model was introduced by Gilbert [6] to model the transmission of radio signals. Roy [16] proved an RSW theorem for vacant crossings in the bounded-radius Poisson blob model, but without the analog of the property that $f(q) \to 1$ as $q \to 1$. As a consequence Roy obtained the equality of various critical points for the Poisson blob model.

Alexander [1] proved RSW Theorem for vacant crossing in a fixed-radius continuum percolation.

In this work we consider a new percolation model on $\mathbb{R}^2$, which is called the Voronoi percolation model on $\mathbb{R}^2$. The Voronoi percolation model has been introduced into the mathematical literature by M.Q. Vahidi-Asl and J.C. Wierman [20], in the context of the first passage percolation. The model we consider was suggested by I. Benjamini and O. Schramm [2], [3].

Voronoi percolation model should be presented as one of a continuum analogues of the discrete percolation model. In Voronoi model we have a Poisson process in $\mathbb{R}^2$ with each Poisson point being the center of the cell of a Voronoi tiling, and for any such cell we say the cell is open with the probability p, or closed with the probability $1-p$, independently of all other cells. The Voronoi percolation model is described in more details in the next chapter.

One can notice that this model has more symmetry then the lattice model. Therefore, it is interesting to check the hypothesis of universality and conformal invariance for this model (see I. Benjamini and O. Schramm [2], [3]). The main purpose of the present work is to estimate the critical value of the parameter p.

Unfortunately, we did not manage to prove the RSW Theorem for the

Voronoi percolation model. The main reason is that in some aspects the structure of our model is much more complex then the structure of the lattice model and the continuum model. The main ideas of the proof of the RSW Theorem for those two models is that we consider the so-called lowest left-right crossing and connect them to the top of the rectangle. In the lattice model we use the fact that the events defined by the points above a certain fixed path γ do not depends on the events defined by the points under the γ. Then, the event that γ is the lowest left-right crossing and an event that there exists an open path from top of the rectangle to path γ (above γ) are independent. This is not true in the Voronoi percolation model. In general for the continuum percolation model this fact is also not true. However, using the boundedness of the radii of the discs, we can "construct" the independence.

One can see, that in the Voronoi percolation model the probability of an event that the diameter of all Voronoi cells in a rectangle is less then some fixed R (R does not depend on the size of a rectangle) tends to zero as the size of a rectangle tends to infinity. That is why we can not base our proof on the same ideas.

We believe that the critical probability for the Voronoi percolation model is equal to $\frac{1}{2}$, but in this paper we prove only the fact that $p_c \geq \frac{1}{2}$, and we will not use RSW Theorem.

The other method is used to prove that $p_c \geq \frac{1}{2}$. It bases on the property of uniqueness. In the third chapter we give basic definitions, and the $0-1$ law for the event that there are exactly k unbounded clusters. We also state the FKG inequality for the Voronoi percolation model. In the fourth chapter we prove the uniqueness theorem for the Voronoi percolation model (in proof of the uniqueness theorem we use ideas from R.M. Burton and M.Keane [4] proof of uniqueness theorem for lattice case). And in the last chapter we prove that $1 > p_c \geq \frac{1}{2}$.

All the facts were consider and proved for the Voronoi percolation model in $\mathbb{R}^2$, but one can see that the theorem 4.1 (uniqueness theorem) and the lemma 5.1 ($p_c < 1$) could be formulated and proved the same way for $\mathbb{R}^d$, where $d > 2$. The method we used for the proof of the theorem 5.1 ($p_c \geq \frac{1}{2}$) does not work in the case of $\mathbb{R}^d$, where $d > 2$.

Chapter 2

Mathematical foundations

Definition 2.1 *Consider a nonempty discrete set Λ in $I\!R^2$. For every point $x \in \Lambda$ let us consider $V(x) = \{z \in I\!R^2 :\mid z - x \mid \leq \mid z - y \mid$, for any $y \in \Lambda$ such that $y \neq x\}$. We call $V(x)$, $x \in \Lambda$ a Voronoi cell and $V(\Lambda) = \{V(x) : x \in \Lambda\}$ a Voronoi tessellation .*

Consider the space $\overline{\Omega}= \{W \in I\!R^2 : W$ is a countable discrete set$\}$. Now for any measurable set $A \subset R^2$, let us define

$$K_n(A) = \left\{\omega \in \Omega : \#\{A \bigcap \omega\} = n\right\}, \text{ where } n = 0, 1, \ldots.$$

Let Θ be the algebra generated by these sets, let $\sigma(\Theta)$ be the smallest σ-algebra containing Θ. Now on $(\overline{\Omega}, \sigma(\Theta))$ we assign the probability measure corresponding to the Poisson process with the density $p\lambda$, where the parameter $p \in [0, 1]$ and $\lambda > 0$ is some fixed number, i.e., $P_p(\cdot)$ has the following properties

1. $P^p(K_n(A)) = \frac{(p\lambda\mu(A))^n}{n!} e^{-p\lambda\mu(A)}$, where μ is the standard Lebesgue measure on $I\!R^2$ ($\mu(A)$ =Vol(A)).

2. $K_n(A), K_m(B)$ are independent events when A, B are disjoint measurable sets, $n, m \subset Z$.

After that we consider the two probability systems $(\Omega_o, \Theta_o, P^p)$ and $(\Omega_c, \Theta_c, P^{1-p})$ where $\Omega_o = \Omega_c = \bar{\Omega}$. Finally, consider the product space $\Omega = \Omega_o \times \Omega_c$ with the σ-algebra F and the product measure P_p.

In our probability space (Ω, F, P_p), $\omega \in \Omega$ is called a *configuration.* Given $\omega = (\omega_o, \omega_c)$, the set ω_o is called the set of open points, and ω_c is called the set of closed points. We say that for some $z \in I\!R^2$ $\omega(z) = 1$, if $z \in \omega_o$;

$\omega(z) = -1$, if $z \in \omega_c$; and $\omega(z) = 0$, if $z \notin \omega$. Finally, we say that $\omega \in \Omega$ gives the Voronoi tessellation $V(\omega) = V(\omega_o \cup \omega_c)$.

Intuitively, $\omega(z) = 1$ implies that there is a Poisson point, and this point is open; $\omega(z) = -1$ implies that the point is closed, and $\omega(z) = 0$ implies that there is no Poisson point.

Let ω, ω' be two configurations in Ω. We say that $\omega \leq \omega'$ if for any $z \in \mathbb{R}^2$ the following inequality holds

$$\omega(z) \leq \omega'(z).$$

Chapter 3

Basic Techniques and Definitions

For any $x \in \omega$ such that $\omega(x) \neq 0$ we say that the cell $V(x)$ is open (closed) if $\omega(x) = 1$ ($\omega(x) = -1$).

Definition 3.1 *A connected component of the union of all open (closed) cells is called an* **open (closed) cluster***.*

Definition 3.2 *Let $\gamma : [0,1] \to I\!R^2$ be a path (continuous curve) in R^2, then we call it an open path if for any $x \in \omega$ such that $V(x) \bigcap \gamma \neq \emptyset$, $V(x)$ is open.*

Definition 3.3 *$x \to y$ (y can be reached from x) if there is an open path γ, such that $\gamma(0) = x$, and $\gamma(1) = y$.*

Definition 3.4 *Let $C_0 = \{y : 0 \to y\}$. If C_0 has infinite diameter then we say that percolation occurs.*

The probability of a percolation is a nondecreasing function of p, so it is natural to define the **critical probability**

$$p_c = \inf\{p : P_p(diam(C_0) = \infty) > 0\}.$$

Definition 3.5 *A random variable N on (Ω, F, P) is called* **increasing** *if $N(\omega) \leq N(\omega')$ whenever $\omega \leq \omega'$; N is called* **decreasing** *if $-N$ is increasing. An event $A \in F$ is called* **increasing** *if I_A (indicator of A) is an increasing random variable.*

As a simple example of increasing events and random variables, consider the event $x \to y$, and a number of points $z = (0, n)$, where n is a natural number, such that $x \to z$.

Theorem 3.1 *If N is an increasing random variable on (Ω, F), then*

$$E_{p_1}(N) \leq E_{p_2}(N) \text{ whenever } p_1 \leq p_2$$

so long as these mean values exist. If A is an increasing event in F, then

$$P_{p_1}(A) \leq P_{p_2}(A) \text{ whenever } p_1 \leq p_2.$$

Proof : (see Grimmett [7]).
Now one can see, that an event $W = \{diam(C_0) = \infty\}$ is an increasing event, then, from the theorem 3.1, we have

$$P_{p_1}(W) \geq P_{p_2}(W), \text{ for any } 1 \geq p_1 \geq p_2 \geq 0,$$

i.e., if $P_{p_1}(W) = 0$, then $P_{p_2}(W) = 0$, for any $p_2 \leq p_1$.

Theorem 3.2 (FKG inequality) *If A and B are both increasing or decreasing events, then*

$$P_p(A \cap B) \geq P_p(A)P_p(B)$$

Proof : (see Roy [16]).

Lemma 3.1 Square root trick : *Let A_1 and A_2 be increasing events with equal probability, then*

$$P(A_1) \geq 1 - \sqrt{1 - P(A_1 \bigcup A_2)}.$$

Proof : Using $P(A_1) = P(A_2)$, we get

$$(1 - P(A_1))^2 = 1 - 2P(A_1) + P(A_1)^2 =$$

$$1 - P(A_1) - P(A_2) + P(A_1)P(A_2) = (*),$$

and using the FKG inequality (A_1, A_2 are increasing events) we obtain

$$(*) \leq 1 - P(A_1) - P(A_2) + P(A_1 \bigcap A_2) = 1 - P(A_1 \bigcup A_2).$$

$\square$

Let A_k be an event that there are exactly k open unbounded clusters. One can see that A_k is a translation invariant, i.e., if $\omega \in A_k$, then $T(\omega) \in A_k$, where T is automorphism of the $I\!R^2$.

Lemma 3.2 *Let A be a translation invariant, measurable event, then*

$$P_p(A) = 0, \ \ or \ P_p(A) = 1,$$

for any $p \in [0,1]$.

Proof : If A is a measurable event, than for any $\varepsilon > 0$ there exists n and an event B_n, which depends only on the state of points from

$$S_n = [-n, n] \times [-n, n],$$

and the probability of symmetric difference of A and B_n is less than ε ,i.e.,

$$P_p(A \triangle B_n) \leq \varepsilon.$$

Let T be an automorphism of the $I\!R^2$ such that $T(S_n) \bigcap S_n = \emptyset$ (e.g., $T(x,y) = (x + 10n, y)$). Then, using invariance of A, we get

$$P_p(A \triangle T(B_n)) = P_p(T(A) \triangle T(B_n)) \leq \varepsilon.$$

Finally, we have

$$P_p(B_n \triangle T(B_n)) \leq P_p((A \triangle T(B_n)) \bigcup (A \triangle B_n)) \leq 2\varepsilon.$$

Now, let $P_p(B_n) = \alpha$. Then, using independence of B_n and $T(B_n)$, we get

$$P_p(B_n \bigcap T(B_n)) = \alpha^2$$

and

$$P_p(B_n \triangle T(B_n)) = 2\alpha - 2\alpha^2.$$

Finally, we have

$$\alpha(1 - \alpha) \leq \varepsilon,$$

i.e., $\alpha \to 1$, or $\alpha \to 0$ as $\varepsilon \to 0$, and

$$P_p(A) = 1 \text{ or } P_p(A) = 0.$$

$\square$

Chapter 4

Uniqueness in Voronoi Percolation

In this chapter we would like to prove that P_p-almost every $\omega \in \Omega$ has at most one unbounded cluster, i.e., that there are no two different unbounded clusters with the probability one. First of all, let us prove that the number of different infinite clusters cannot be infinite.

Let $N = \#\{x \in \omega : V(x) \cap [-\frac{1}{2}, \frac{1}{2}] \neq \emptyset\}$.

Lemma 4.1

$$E_p N < \infty.$$

Proof : Let $D_i = \{x \in R^2 : |x| \leq i\}$, and

$$A_1 = D_1,$$

$$A_i = D_i \backslash D_{i-1} \text{ for } i = 2, 3, \ldots.$$

Assume that there is at least one point from ω in A_i, i.e., $\omega \cap A_i \neq \emptyset$, then, one can see, that an event $N = k$ depends only on the state of points from $D_{i+\frac{1}{2}}$ (see Figure 6.1). Let $O_1 = \Omega$,

$$O_i = \{\omega : \omega \cap D_{i-1} = \emptyset\}, \text{ for } i = 2, 3, \ldots$$

and

$$S_i = \{\omega : \omega \cap A_i \neq \emptyset\}, \text{ for } i = 1, 2, \ldots.$$

Finally, let $Q_i = O_i \bigcap S_i$ and $N_i = \#\{\omega \cap (D_{i+1} \backslash D_{i-1})\}$. Then, we have

$$E_p(N|Q_i) \leq E_p(N_i|Q_i),$$

because for any $x \in \omega$ ($\omega(x) \neq 0$), where $\omega \in Q_i$ and $V(x) \cap [-\frac{1}{2}, \frac{1}{2}] \neq \emptyset$, we have that $x \in D_{i+1} \backslash D_{i-1}$.
Now

$$E_p(N_i|Q_i) = \sum_{k=1}^{\infty} kP_p(N_i = k|Q_i) = \sum_{k=1}^{\infty} k \frac{P_p((N_i = k) \bigcap O_i \bigcap S_i)}{P_p(Q_i)}$$

$$\leq \frac{1}{P_p(Q_i)} \sum_{k=1}^{\infty} kP_p((N_i = k))P_p(O_i) = \frac{P_p(O_i)}{P_p(Q_i)} E_p N_i = \frac{P_p(O_i)}{P_p(Q_i)} \lambda\pi 4i.$$

And, finally,

$$E_p N = \sum_{i=1}^{\infty} P(Q_i) E_p(N|Q_i) \leq \sum_{i=1}^{\infty} P(Q_i) E_p(N_i|Q_i) \leq$$

$$\sum_{i=1}^{\infty} P_p(O_i) \lambda\pi 4i \leq 4\lambda\pi \sum_{i=1}^{\infty} ie^{-\lambda\pi(i-1)^2} < \infty.$$

$\square$

Lemma 4.2

$$P(A_\infty) = 0.$$

Proof : We say that a point $z \in I\!R^2$ is an **encounter** point for $\omega \in \Omega$ if the following conditions hold:

1. z belongs to an unbounded open cluster C in ω, $\omega(z) = 1$,
2. the set $C \backslash V(z)$ has exactly three unbounded components,
3. $\#\{x : \omega(x) = 1, V(x) \bigcap V(z) \neq \emptyset, x \neq z\} = 3$.

Preposition 4.1 *Let $S_{\frac{1}{2}} = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, and let G be an event that there is encounter point $z \in S_{\frac{1}{2}}$. If $P_p(A_\infty) = 1$, then*

$$P_p(G) > 0.$$

Proof : Let Q_n be an event that there is at least 3 different unbounded open clusters intersecting $S_n = [-n, n] \times [-n, n]$. If $P_p(A_\infty) = 1$, then there exists n such that $P_p(Q_n) > 0$. Let us define $\omega_b = \omega \bigcap S_n$ and $\omega_a = \omega \backslash \omega_b$ (see Figure 6.2), for any $\omega \in \Omega$. The correspondence $\omega \leftrightarrow (\omega_a, \omega_b)$ induces a product structure $\Omega = \Omega_a \times \Omega_b$. Let

$$\Phi = \{\omega_a \in \Omega_a : P_p(\omega_b \in \Omega_b : \omega_a \cup \omega_b \in Q_n) > 0\}.$$

Next, using Fubini theorem, we have

$$P_p(Q_n\backslash(\Phi \times \Omega_b)) = \int\limits_{\omega_a \in (\Omega_a \backslash \Phi)} P_p(\omega_b \in \Omega_b : \omega_a \cup \omega_b \in Q_n) d\omega_a = 0.$$

Then,

$$P_p(\Phi) = P_p(\Phi \times \Omega_b) \geq P_p(Q_n) > 0.$$

Now let us fix $\omega_a \in \Phi$. Then, there exist at least three unbounded open clusters outside S_n starting "close" to ∂S_n (see Figure 6.2). Consider any three of them. We can construct a continuation of each of these three clusters to $S_{\frac{1}{2}}$ using only the points from S_n. The three unbounded open clusters we obtain shall intersect only in one point $z \in S_{\frac{1}{2}}$ (see Figure 6.2). The probability of such construction may be small, but it is greater then zero. We have, that for fixed $\omega_a \in \Phi$

$$P_p(\omega_b \in \Omega_b : \omega_a \cup \omega_b \in G) > 0.$$

Now we can use Fubini theorem and obtain that

$$P_p(G) = \int\limits_{\omega_a \in \Omega_a} P_p(\omega_b \in \Omega_b : \omega_a \cup \omega_b \in G) d\omega_a \geq$$

$$\int\limits_{\omega_a \in \Phi} P_p(\omega_b \in \Omega_b : \omega_a \cup \omega_b \in G) d\omega_a > 0,$$

because we integrate the function that is more then 0 over the set with the non-zero measure.

$\square$

Now let us consider a square S_L ($L \in N, L >> 1$). Let $k = 2l$, then we divide S_L in squares 1×1 (see Figure 6.4). Let X be the number of the squares with at least one encounter point, then on the one hand

$$E_p X = \sum_{i=1}^{k^2} P_p(G) \geq \alpha k^2,$$

where $\alpha > 0$. On the other hand,

$$E_p X \leq k^2 P_p(X \geq \frac{\alpha k^2}{2}) + \frac{\alpha k^2}{2} P_p(X < \frac{\alpha k^2}{2}).$$

We have

$$\alpha \leq P_p(X \geq \frac{\alpha k^2}{2}) + \frac{\alpha}{2}(1 - P_p(X \geq \frac{\alpha k^2}{2})),$$

and, finally,

$$P_p(X \geq \frac{\alpha k^2}{2}) \geq \frac{\alpha}{2-\alpha}.$$

Now let we have m encounter points in S_l. Let x be an encounter point. Then, here exists an open cluster C_x such that $V(x) \in C_x$ and $C_x \backslash V(x)$ has exactly three $(C_x^1, C_x^2, C_x^2\ )$ components. Let $y \neq x$ be an encounter point, $V(y) \subset C_x^i$ for some $i \in \{1,2,3\}$. Using the fact that $C_x^i \bigcap C_x^j = \emptyset$ for any $j \neq i$, $j \in \{1,2,3\}$, we have $V(y) \bigcap C_x^j = \emptyset$, i.e., clusters cannot form a "loop" with encounter points on it. Otherwise, there exists an encounter point for which one of the conditions (2) and (3) is violated. Then, the unbounded open clusters starting in those points form a "forest" (see Figure 6.4). Therefore, the number of branches of the forest is at least the number of vertexes, i.e., m. Then we have, that $\#\{z : V(z) \bigcap \partial S_L \neq \emptyset, \omega(z) = 1\} \geq m$, and using the fact that all branches are different, we have

$$\#\{z : V(z) \bigcap \partial S_L \neq \emptyset, \omega(z) \neq 0\} \geq m.$$

Let

$$Y_L = \#\{z : V(z) \bigcap \partial S_L \neq \emptyset, \omega(z) \neq 0\},$$

then

$$P_p(Y_L \geq 2\alpha L^2) \geq \frac{\alpha}{2-\alpha} > 0.$$

From above inequality it follows that $E_p Y_L \geq CL^2$, where $C > 0$. Then, from lemma 4.1 it follows, that $E_p Y_L \leq 4cL$, i.e., we have a **contradiction**. Therefore, $P(A_\infty) = 0$.

□

Lemma 4.3

$$P_p(A_k) = 0,$$

where $k = 2, 3, \ldots$.

Proof : Let $P_p(A_k) > 0$ for some fixed $k \in 2, 3, \ldots$, then (using lemma 3.2) $P_p(A_k) = 1$, and $P_p(A_1) = 0$. Now if $P_p(A_k) = 1$, then there exists n such, that with the probability more then zero, there are exactly k different unbounded open clusters intersecting S_n. Thus, we can use the same method as in the proof of the proposition 4.1 (see Figure 6.3) and show that

$$P_p(A_1) > 0,$$

i.e., we come to contradiction. Therefore, $P_p(A_k)$ should be zero for $k = 2, 3, \ldots$.

$\square$

And, finally, from lemma 4.2 and lemma 4.3 follows the uniqueness theorem.

Theorem 4.1 *Let $p \in [0,1]$, then P_p-almost every $\omega \in \Omega$ has at most one infinite cluster.*

Chapter 5

Critical phenomena, $1 > p_c \geq \frac{1}{2}$

Lemma 5.1 *There exists $p < 1$, such that*

$$P_p(diam(C_0) = \infty) > 0,$$

i.e.,

$$p_c < 1.$$

Proof : Let us partition $I\!R^2$ by squares with the side length δ (we shall choose δ). We call the square S open if the following conditions hold:

1. There is no closed point in S, i.e., $\omega(z) \geq 0$, for $z \in S$,
2. We divide S into nine equal squares, and then there is at least one open point in each of them.

We call the square S closed if it is not open. Let S_o be an event that S is open, then we have

$$P_p(S_o) \geq e^{-\lambda(1-p)\delta^2}(1 - e^{-\lambda p(\frac{\delta}{3})^2})^9.$$

Let $\delta^2 = \frac{9h}{\lambda}$, where $h > 0$, then

$$P_p(S_o) \geq e^{-9h(1-p)}(1 - e^{-hp})^9.$$

One can see, that if we have an unbounded cluster of open squares, then we have an unbounded open cluster in our Voronoi percolation model. We know, that the critical probability for site percolation in Z^2 is less then one (see Kesten [12]), i.e., it is equal to some $l < 1$, then we can find $p < 1$, and $h > 0$ such that $P_p(S_o) \geq l$. Also, different squares are independently open or closed. Identifying the squares with the vertexes of the lattice, we see that the union of all open squares contains an unbounded open cluster.

$\square$

Definition 5.1 *An* **open left-right** *(respectively* **closed top-bottom crossing** *of the rectangle B is an open (respectively closed) path $\gamma : [0,1] \to B$, which joins some point from the left (respectively lower) side of B, i.e., $\gamma(0) \in$ left (respectively lower) side, to some point from the right (respectively upper) side of B, i.e., $\gamma(1) \in$ right (respectly upper).*

We write $LR_o(B)$ (respectively TB_c) for the event that there is an open left-right (respectively closed top-bottom) crossing of B, and let D_B be an event that one of following two events occurs

1. $z_1, z_2, z_3, z_4 \in \omega : V(z_1) \bigcap V(z_2) \bigcap V(z_3) \bigcap V(z_4) \bigcap B \neq \emptyset$,
2. $x_1, x_2, x_3 \in \omega : V(x_1) \bigcap V(x_2) \bigcap V(x_3) \bigcap \partial B \neq \emptyset$.

If $\omega \in LR_o(B) \bigcap TB_c(B)$, then there should to be a point in B where top-bottom closed crossing intersect left-right crossing (see Figure 6.5). If this point is not in the boundary of B, then (1) occurs; if it is in the boundary, then (2) occurs, i.e.,

$$LR_o(B) \bigcap TB_c(B) \subset D_B.$$

From this it follows, that

$$P_p(LR_o(B) \bigcap TB_c(B)) \leq P_p(D_B) = 0.$$

Theorem 5.1 *There is no percolation when $p = \frac{1}{2}$, i.e., the critical probability of the percolation on the Voronoi tessellation is greater then or equal to $\frac{1}{2}$.*

Proof : The argument is due to Zhang and it appears in Grimmett [7]. Let W_o (respectively W_c) be an event that there is an unbounded open (respectively closed) cluster, and suppose $P_{\frac{1}{2}}(W_o) = P_{\frac{1}{2}}(W_c) > 0$. Using lemma 3.2, we have that

$$P_{\frac{1}{2}}(W_o) = P_{\frac{1}{2}}(W_c) = 1.$$

Then, for a big enough box S_n the probability that unbounded open cluster intersects S_n is almost equal to one. Let W_n be an event that there is unbounded open cluster intersecting S_n. Let O_1^n (O_2^n) be an event that there is the infinite open path γ starting from the left or the top (the right or the bottom) side of S_n which does not intersect S_n (i.e., there exists an infinite open path γ such that $\#\{z \in \omega : V(z) \bigcap S_n \neq \emptyset, V(z) \bigcap \gamma \neq \emptyset\} = 1$), then O_1^n,

O_2^n are increasing events, and $P_p(O_1^n) = P_p(O_2^n)$. Moreover, $O_1^n \bigcap O_2^n = W_n$. Therefore, applying the square root trick (see lemma 3.1),

$$\lim_{n \to \infty} P_p(O_1^n) = 1.$$

Next, let O_l^n (O_t^n) be an event that there is the infinite open path γ starting from the left (the top) side of S_n which does not intersect S_n. Then, applying the square root trick (see lemma 3.1) to events O_l^n, O_t^n, we have

$$\lim_{n \to \infty} P_p(O_l^n) = 1.$$

Finally, let us consider an event that two unbounded open clusters start at the left side and the right side of S_n, and two unbounded closed cluster start at the top and at the bottom of S_n (see Figure 6.6). By the uniqueness theorem, there exist at most one unbounded open cluster and at most one unbounded closed cluster. Therefore, there should exists the left-right open crossing of S_n and the top-bottom closed crossing of S_n. However, the probability of such an event is zero. Therefore, $P_{\frac{1}{2}}(W_o)$ should be zero.

$\square$

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Chapter 6

Figures

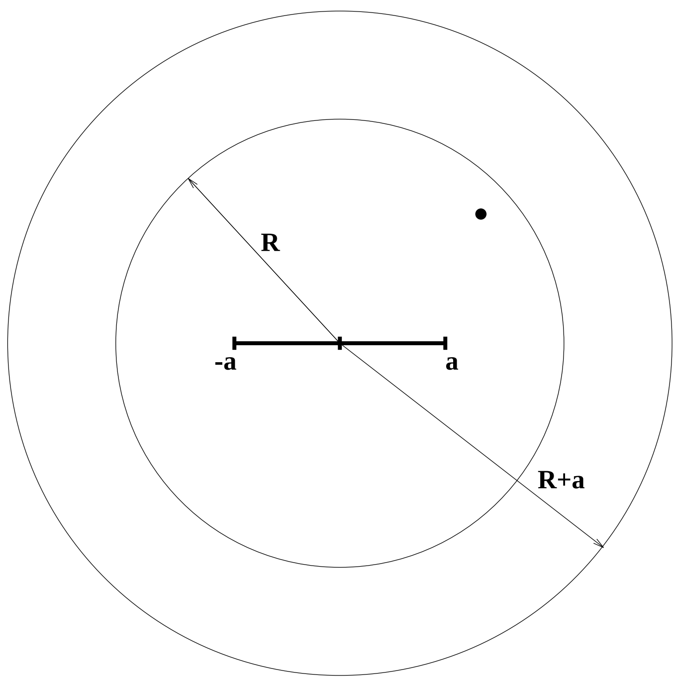

Figure 6.1: Let $Z = \#\{\omega \cap [-a, a]\}$. Assume that there is at least one point in the disc $D_1 = \{x \in I\!R^2 : |x| \leq R\}$, then $V(x) \cap [-a, a] = \emptyset$, for any $x : |x| > a + R$.

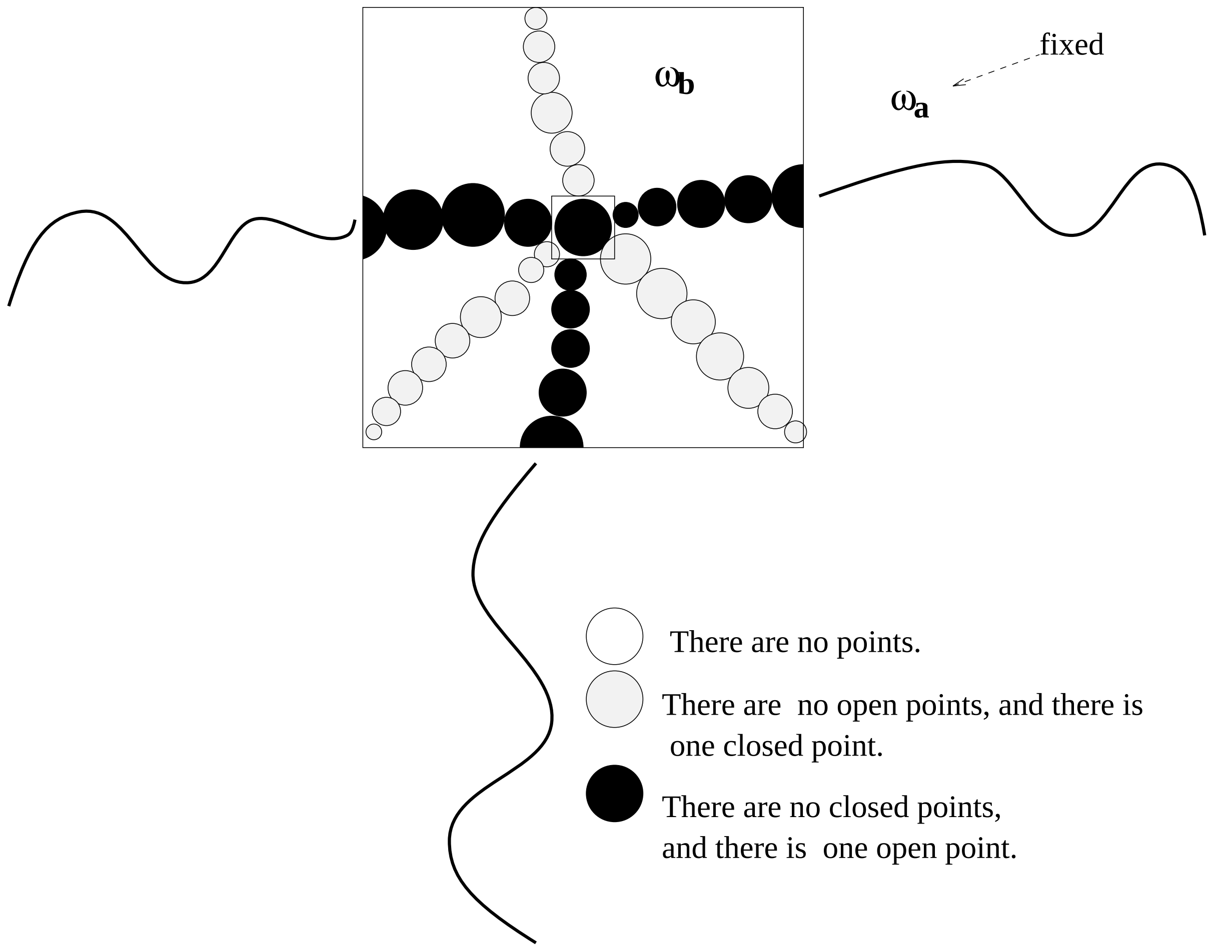

Figure 6.2: The probability of such an event is very small, but more than zero.

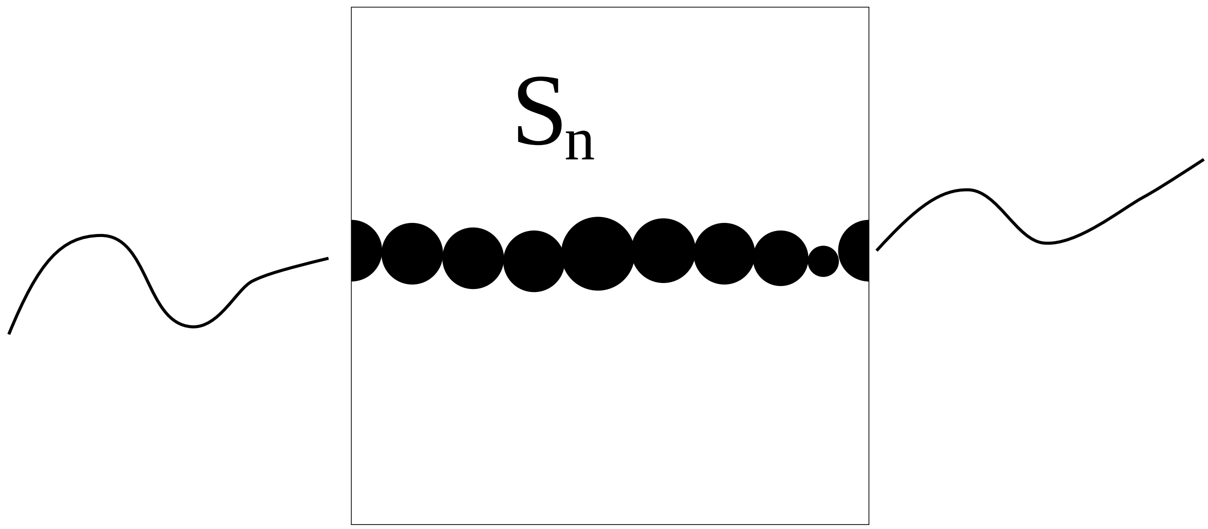

Figure 6.3: Such an event a has very small probability, but it is more than zero.

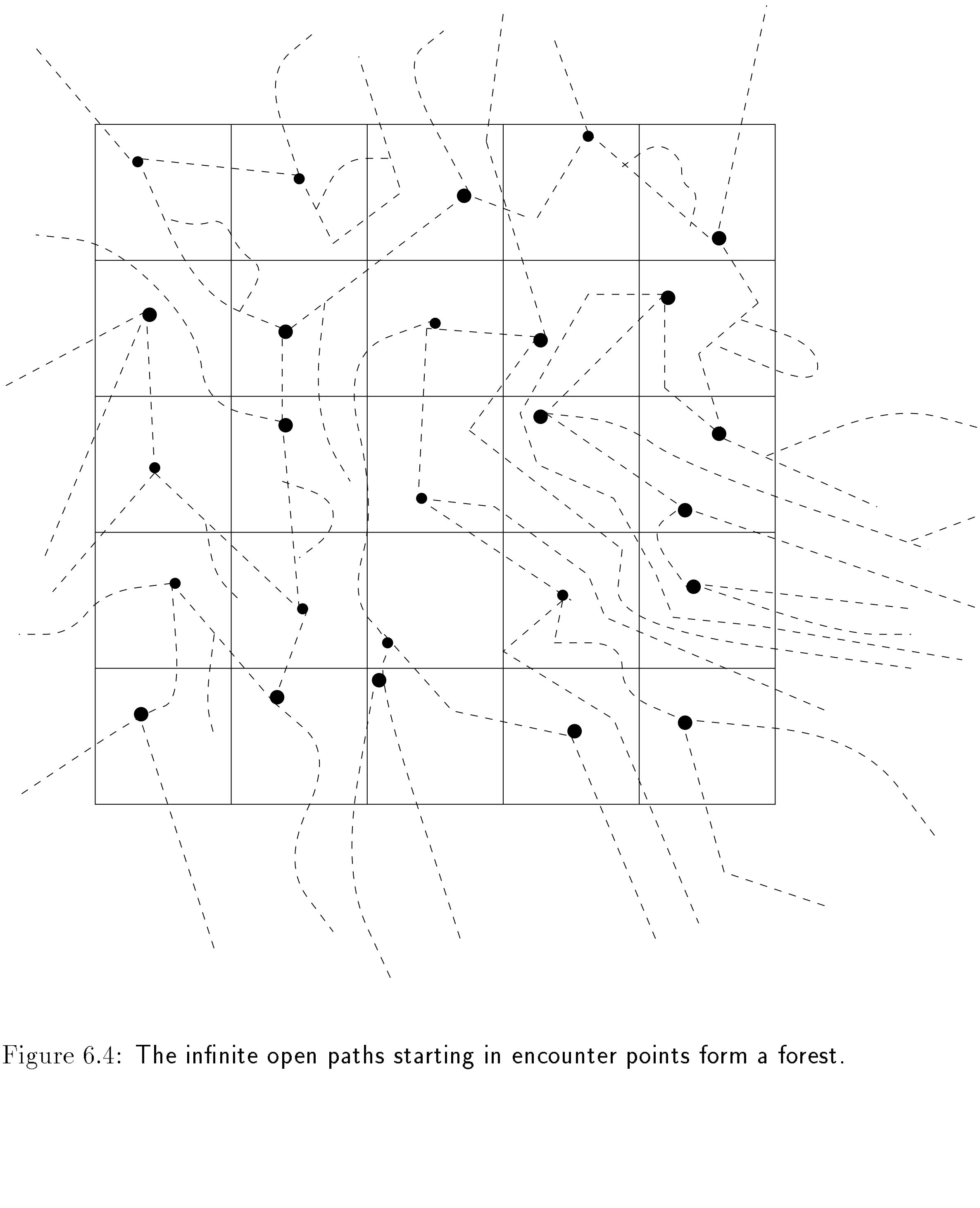

Figure 6.4: The infinite open paths starting in encounter points form a forest.

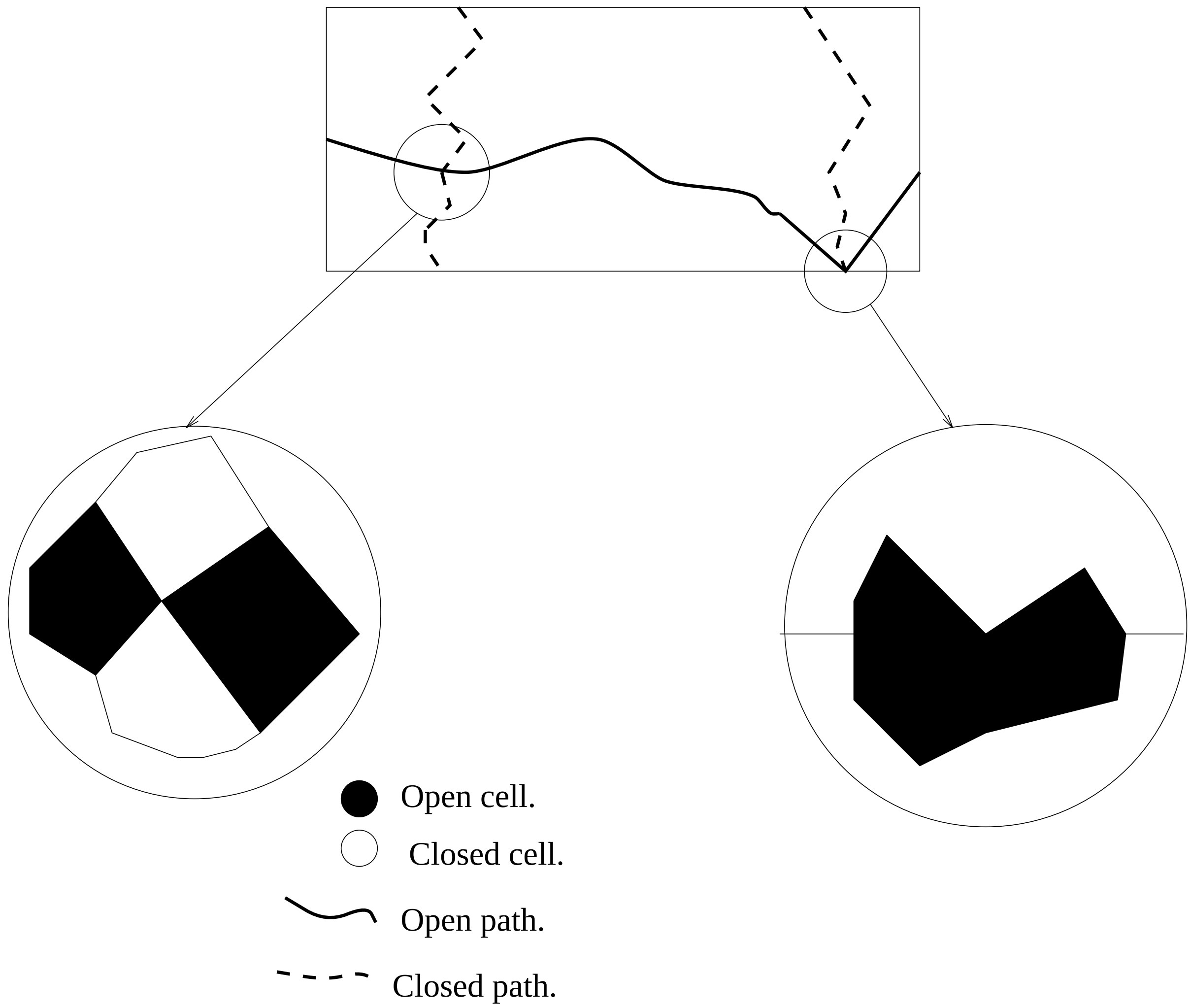

Figure 6.5: An event that the open path intersects the closed path.

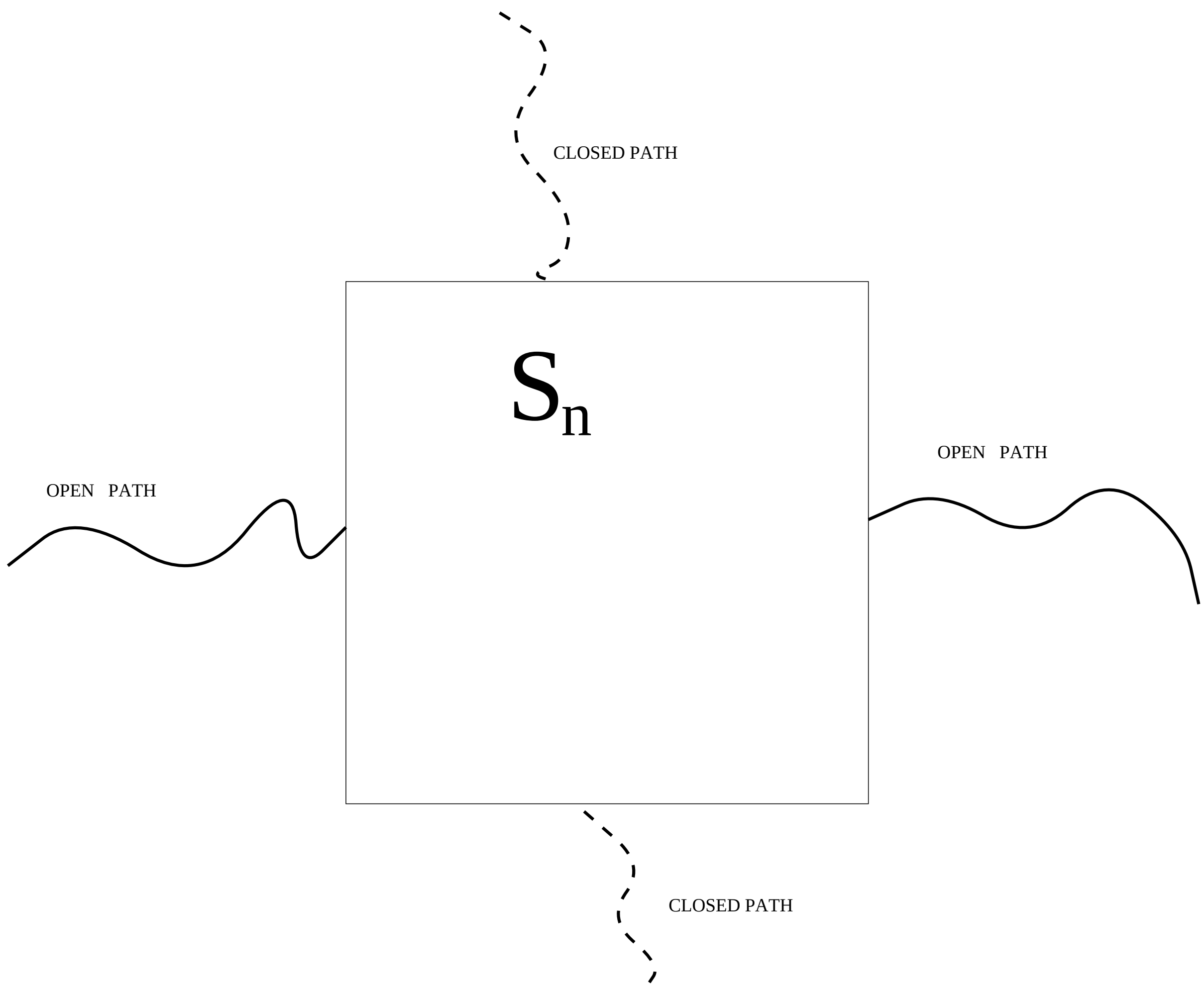

Figure 6.6: It is impossible to connect the two open infinite paths and the two closed infinite paths at the same time .