TOPICS IN PROBABILITY THEORY AND
STOCHASTIC PROCESSES
Home Work 1, due on THURSDAY SEPTEMBER 4,
Instructor: Prof. Artem Zvavitch

Problem 1. If three fair dice are tossed, what is the probability that the sum is 6? What is the probability that one of the dice shows 1 given that the sum of all three is 6?

Problem 2. If three fair dice are tossed, what is the average number of the sum? What about 781 dice?

Problem 3. Suppose $X$ is a random variable such that $P(X = -2) = P(X = -1) = \frac{1}{6}$, $P(X = 0) = \frac{1}{2}$ and $P(X = 2) = P(X = 1) = \frac{1}{12}$.

Please, find $E[X]$, $E[X^2]$ and $\text{Var} X = E[X - E[X]]^2$. Assume $Y$ is an independent copy of $X$, please, find $E[3X + 6Y]$ and $E[XY]$. Please, also compute $E[X|X + Y = 0]$.

Problem 4. Assume $X$ is a uniform random variable on the interval $[-1, 1]$ (i.e. $X$ has a density function $f(x) = \frac{1}{2}$, for $x \in [-1, 1]$ and $f(x) = 0$ otherwise). Please, find cumulative distribution function $F(x) = P(X \leq x)$, $E[X]$, $E[X^2]$ and $\text{Var} X$.

Problem 5. The joint density of random variables $X$ and $Y$ is given by

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y}, \text{ where } 0 < x < \infty, 0 < y < \infty.$$  

Compute $E[X|Y = y]$.

Problem 6. Let $X_i, i \geq 0$ be independent and identically distributed random variable with probability mass function $p(k) = P(X_i = k)$, for $k = 1, \cdots, m$ (and $\sum_{j=1}^{m} p(j) = 1$).

Find $E[N]$, where $N = \min\{n : X_n = X_0\}$. (Hint: use conditional expectation!)

Problem 7. (5 bonus points) Assume $X$ and $Y$ are a uniform random variables on the interval $[-1, 1]$. Find $E[X|X + Y = y]$. 

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\caption{Figure Caption}
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