

# PROJECTIONS OF CONVEX BODIES AND THE FOURIER TRANSFORM

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ABSTRACT. The Fourier analytic approach to sections of convex bodies has recently been developed and has led to several results, including a complete analytic solution to the Busemann-Petty problem, characterizations of intersection bodies, extremal sections of  $l_p$ -balls. In this article, we extend this approach to projections of convex bodies and show that the projection counterparts of the results mentioned above can be proved using similar methods. In particular, we present a Fourier analytic proof of the recent result of Barthe and Naor on extremal projections of  $l_p$ -balls, and give a Fourier analytic solution to Shephard's problem, originally solved by Petty and Schneider and asking whether symmetric convex bodies with smaller hyperplane projections necessarily have smaller volume. The proofs are based on a formula expressing the volume of hyperplane projections in terms of the Fourier transform of the curvature function.

## 1. INTRODUCTION

The study of geometric properties of bodies based on information about sections and projections of these bodies has important applications to many areas of mathematics and science. A new approach to sections of convex bodies, based on methods of Fourier analysis, has recently been developed and has led to several results including an analytic solution to the Busemann-Petty problem (see the survey [K4] for a brief description of this approach). The goal of this article is to extend this method to projections of convex bodies and show that several important results on projections can be proved in the same spirit as their section counterparts.

The Fourier transform approach to sections of convex bodies is based on certain formulas connecting the volume of sections with the Fourier transform of powers of the Minkowski functional. For instance, it was proved in [K5] that, for any symmetric convex body  $K$  in  $\mathbb{R}^n$  and every  $\xi \in S^{n-1}$ ,

$$\text{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi(n-1)} (\|\cdot\|_K^{-n+1})^\wedge(\xi), \quad (1)$$

where  $\|x\|_K = \min\{a \geq 0 : x \in aK\}$  is the Minkowski functional of  $K$ ,  $\xi^\perp$  is the central hyperplane orthogonal to  $\xi$ , and the Fourier transform is considered in the sense of distributions.

In Section 2 we prove the projection analog of this formula (see Theorem 2 below): if the surface area measure of  $K$  is absolutely continuous, then

$$\text{Vol}_{n-1}\left(K\Big|_{\theta^\perp}\right) = -\frac{1}{\pi}\widehat{f_K}(\theta) \quad \forall \theta \in S^{n-1}, \quad (2)$$

where  $K\Big|_{\theta^\perp}$  is the orthogonal projection of  $K$  onto the hyperplane  $\theta^\perp$ ,  $f_K$  is the curvature function of the body  $K$  extended to a homogeneous of degree  $-n-1$  function on  $\mathbb{R}^n$ , and the Fourier transform is in the sense of distributions.

In Section 3 we apply formula (2) to give a Fourier analytic proof of the recent result of Barthe and Naor [BN] that, for  $p \geq 2$ , the minimal and maximal hyperplane projections of the unit balls  $B_p^n$  of the spaces  $l_p^n$  are the ones corresponding to the vectors  $\theta_1 = (1, 0, \dots, 0)$  and  $\theta_n = (1/\sqrt{n}, \dots, 1/\sqrt{n})$ , respectively. The proof in [BN] is based on a certain formula for the volume of projections (see formula (8)) that is obtained using probabilistic arguments. We show that this formula is a particular case of the formula (2) and can be derived by a direct computation of the Fourier transform of the curvature function of  $l_p$ -balls. This makes the proof of the result of Barthe and Naor similar to the proof of the corresponding result for hyperplane sections of the balls  $B_p^n$  with  $0 < p \leq 2$  in [K5].

Another application is to the Shephard problem. The problem (see [Sh]) reads as follows. Let  $K, L$  be convex symmetric bodies in  $\mathbb{R}^n$  and suppose that, for every  $\theta \in S^{n-1}$ ,

$$\text{Vol}_{n-1}\left(K\Big|_{\theta^\perp}\right) \leq \text{Vol}_{n-1}\left(L\Big|_{\theta^\perp}\right). \quad (3)$$

Does it follow that

$$\text{Vol}_n(K) \leq \text{Vol}_n(L)? \quad (4)$$

The problem was solved independently by Petty [P] and Schneider [Sc1], who showed that the answer is affirmative if  $n \leq 2$  and negative if  $n \geq 3$ . Further results were obtained by Ball [B], who proved that it is necessary to multiply  $\text{Vol}_n(L)$  by  $\sqrt{n}$  to make the answer affirmative in all dimensions, and by Goodey and Zhang [GZ], who solved the generalized Shephard's problem with lower dimensional projections. The section counterpart of Shephard's problem is the Busemann-Petty problem asking whether symmetric convex bodies with larger central hyperplane sections necessarily have greater volume. The solution to

the Busemann-Petty problem has recently been completed (see [GKS] and [Zh] for historical details) and the answer is affirmative if  $n \leq 4$  and negative if  $n \geq 5$ .

In Section 5, we present a new solution of the Shephard problem, which is completely Fourier analytic, and, along with formula (2), is based on a certain spherical version of Parseval's formula. This solution is in the spirit of the solution to the Busemann-Petty problem from [K3].

In Section 4, we give a Fourier analytic characterization of projection bodies, which, in particular, leads to the following connection between projections and sections: if  $n$  is an even integer and  $L$  is a symmetric convex body in  $\mathbb{R}^n$  with infinitely smooth (on  $S^{n-1}$ ) support function, then  $L$  is a projection body if and only if, for every  $\xi \in S^{n-1}$ , we have  $(-1)^{n/2} A_{L^*, \xi}^{(n)}(0) \geq 0$ , where  $A_{L^*, \xi}$  is the parallel section function of the polar body  $L^*$  in the direction of  $\xi$ . If  $n$  is odd the condition is slightly more complicated, but is also expressed in terms of sections. These characterizations of projection bodies are similar to that of intersection bodies in [K6]. In particular, Theorem 1 from [K6], in conjunction with the formula (15), implies that an infinitely smooth symmetric convex body  $L$  in  $\mathbb{R}^n$  (with even  $n$ ) is an intersection body if and only if, for every  $\xi \in S^{n-1}$ ,  $(-1)^{(n-2)/2} A_{L, \xi}^{(n-2)}(0) \geq 0$ . These conditions explain why the transition in Shephard's problem occurs between the dimensions 2 and 3, while in the Busemann-Petty problem it happens between 4 and 5: convexity allows to control the derivatives only up to the second order.

## 2. VOLUME OF PROJECTIONS VIA FOURIER TRANSFORM.

Our main tool is the Fourier transform of distributions. We denote by  $\mathcal{S}$  the space of rapidly decreasing infinitely differentiable functions (test functions) on  $\mathbb{R}^n$  with values in  $\mathbb{C}$ . By  $\mathcal{S}'$  we denote the space of distributions over  $\mathcal{S}$ . Every locally integrable real valued function  $f$  on  $\mathbb{R}^n$  with power growth at infinity represents a distribution acting by integration: for every  $\phi \in \mathcal{S}$ ,  $\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx$ . The Fourier transform of a distribution  $f$  is defined by  $\langle \hat{f}, \hat{\phi} \rangle = (2\pi)^n \langle f, \phi \rangle$ , for every test function  $\phi$ .

Let  $\mu$  be a finite Borel measure on the unit sphere  $S^{n-1}$ . We extend  $\mu$  to a homogeneous distribution of degree  $-n-1$ . A distribution  $\mu_e$  is called the extended measure of  $\mu$  if, for every even test function

$\phi \in \mathcal{S}(\mathbb{R}^n)$ ,

$$\langle \mu_e, \phi \rangle = \frac{1}{2} \int_{S^{n-1}} \langle r^{-2}, \phi(r\xi) \rangle d\mu(\xi). \quad (5)$$

In most cases we are only interested in test functions supported outside of the origin, for which  $\langle r^{-2}, \phi(r\xi) \rangle = \int_{\mathbb{R}} r^{-2} \phi(r\xi) dr$ .

If  $\mu$  is absolutely continuous with the density  $g \in L_1(S^{n-1})$ , we define the extension  $g(x)$ ,  $x \in \mathbb{R}^n \setminus \{0\}$  as a homogeneous function of degree  $-n-1$ :  $g(x) = |x|^{-n-1} g(x/|x|)$  and identify  $\widehat{\mu}_e$  with  $\widehat{g}$ .

Throughout the paper, we write that two homogeneous distributions are equal on the sphere meaning that their homogeneous extensions are equal as distributions on  $\mathbb{R}^n$ . For instance, the distributions in both sides of (1) are homogeneous of degree -1, while in (2) the degree of homogeneity is 1. Recall that the Fourier transform of an even homogeneous distribution of degree  $p$  is an even homogeneous distribution of degree  $-n-p$ .

The following fact is well-known, see for example [Se], formula 4.7. We include here a proof similar to that from [K2], Lemma 2.

**Theorem 1.** *For every  $\theta \in S^{n-1}$ ,*

$$\widehat{\mu}_e(\theta) = -\frac{\pi}{2} \int_{S^{n-1}} |(\theta, y)| d\mu(y).$$

**Proof :** Let  $\phi \in \mathcal{S}(\mathbb{R}^n)$  be an even test function so that  $0 \notin \text{supp}(\widehat{\phi})$ . Then, by the definition of  $\mu_e$ ,

$$\langle \widehat{\mu}_e, \phi \rangle = \langle \mu_e, \widehat{\phi} \rangle = \frac{1}{2} \int_{S^{n-1}} d\mu_e(\theta) \int_{-\infty}^{\infty} r^{-2} \widehat{\phi}(r\theta) dr.$$

By Lemma 2.1 from [K2]

$$\int_{\mathbb{R}^n} |(\theta, x)| \widehat{\psi}(x) dx = (2\pi)^{n-1} \frac{4\sqrt{\pi}}{\Gamma(-1/2)} \int_{\mathbb{R}} r^{-2} \psi(r\theta) dr$$

for any even  $\psi \in \mathcal{S}(\mathbb{R}^n)$  with  $0 \notin \text{supp}(\psi)$ .

This gives (with  $\widehat{\psi} = \phi$ )

$$\begin{aligned} \langle \widehat{\mu}_e, \phi \rangle &= -\frac{\pi}{2} \int_{S^{n-1}} d\mu(\theta) \int_{\mathbb{R}^n} |(\theta, \xi)| \phi(\xi) d\xi = \\ &= -\frac{\pi}{2} \int_{\mathbb{R}^n} \phi(\xi) d\xi \int_{S^{n-1}} |(\theta, \xi)| d\mu(\theta) = \left\langle -\frac{\pi}{2} \int_{S^{n-1}} |(\theta, \xi)| d\mu(\theta), \phi \right\rangle. \end{aligned}$$

Since  $\phi$  is an arbitrary even test function with  $0 \notin \text{supp}(\widehat{\phi})$ , the distributions  $\widehat{\mu}_e$  and  $-\pi/2 \int_{S^{n-1}} |(\theta, \xi)| d\mu(\theta)$  can differ by a polynomial only. But both distributions are even and homogeneous of degree one, so the polynomial must be equal to zero, and  $\widehat{\mu}_e$  coincides with the continuous function  $-\pi/2 \int_{S^{n-1}} |(\theta, \xi)| d\mu(\theta)$ .

□

To apply Theorem 1 to the study of volumes of projections of convex bodies, we use the well-known Cauchy formula ([G], page 361):

$$\text{Vol}_{n-1} \left( L \Big| \theta^\perp \right) = \frac{1}{2} \int_{S^{n-1}} |\theta \cdot v| dS_{n-1}(L, v), \quad \theta \in S^{n-1}. \quad (6)$$

Here  $S_{n-1}(L, \cdot)$  is the surface area measure of  $L$  ([G], page 351).

A convex body  $L$  is said to have a curvature function

$$f_L(\cdot) : S^{n-1} \rightarrow \mathbb{R},$$

if its surface area measure  $S_{n-1}(L, \cdot)$  is absolutely continuous with respect to Lebesgue measure  $\sigma_{n-1}$  on  $S^{n-1}$ , and

$$\frac{dS_{n-1}(L, \cdot)}{d\sigma_{n-1}} = f_L(\cdot) \in L_1(S^{n-1}).$$

The next statement follows from the Cauchy formula (6) and Theorem 1. We denote by  $S_e(L)$  the extended measure of the surface area measure  $S_{n-1}(L, \cdot)$ .

**Theorem 2.** *Let  $L$  be a convex origin symmetric body in  $\mathbb{R}^n$ . Then*

$$\widehat{S_e(L)}(\theta) = -\pi \text{Vol}_{n-1} \left( L \Big| \theta^\perp \right), \quad \forall \theta \in S^{n-1}.$$

*In particular, if the body  $L$  has a curvature function then*

$$\widehat{f_L}(\theta) = -\pi \text{Vol}_{n-1} \left( L \Big| \theta^\perp \right), \quad \forall \theta \in S^{n-1}.$$

### 3. PROJECTIONS OF THE UNIT BALL OF $\ell_p^n$

In order to show similarities between the section and projection cases, we first recall how was formula (1) applied in [K5] to the problem of finding the extremal sections of the unit balls  $B_p^n$  of the spaces  $l_p^n$ ,  $0 < p \leq 2$ . The result of [K5] is as follows: if  $0 < p \leq 2$  then, for every  $\theta \in S^{n-1}$ ,

$$\text{Vol}_{n-1} (B_p^n \cap \theta_n^\perp) \leq \text{Vol}_{n-1} (B_p^n \cap \theta^\perp) \leq \text{Vol}_{n-1} (B_p^n \cap \theta_1^\perp),$$

where  $\theta_1 = (1, 0, \dots, 0)$  and  $\theta_n = (1/\sqrt{n}, \dots, 1/\sqrt{n})$ .

In the case of  $l_p$ -balls the Fourier transform in the right-hand side of (1) can be expressed in terms of the function  $\gamma_p$  which is the Fourier

transform of the function  $z \rightarrow \exp(-|z|^p)$ ,  $z \in \mathbb{R}$ . The formula (1) turns into

$$\text{Vol}_{n-1}(B_p \cap \xi^\perp) = \frac{p}{\pi(n-1)\Gamma((n-1)/p)} \int_0^\infty \prod_{k=1}^n \gamma_p(t\xi_k) dt, \quad (7)$$

which is true for every  $p > 0$  and  $\xi \in S^{n-1}$ . Note that the latter formula was first proved by Meyer and Pajor [MP] for  $1 \leq p \leq 2$  using probabilistic methods, and that it was used in [MP] to find the extremal sections of  $B_1^n$ .

The second step in [K5] is to prove that, for  $0 < p \leq 2$ , the function  $\gamma_p(\sqrt{\cdot})$  is log-convex on  $[0, \infty)$ . The proof is based on the fact that the function  $\exp(-|\cdot|^\alpha)$  is completely monotonic on  $[0, \infty)$  for every  $\alpha \in (0, 1]$ .

The log-convexity implies that for any choice of numbers  $0 < \xi_1 < \eta_1 < \eta_2 < \xi_2$  with  $\xi_1^2 + \xi_2^2 = \eta_1^2 + \eta_2^2 = 1$  and any  $t > 0$ , one has

$$\gamma_p(t\xi_1)\gamma_p(t\xi_2) \geq \gamma_p(t\eta_1)\gamma_p(t\eta_2).$$

It is now clear that the integrand in the right-hand side of (7) is minimal if all the coordinates of  $\xi$  are equal, and maximal when  $\xi$  has only one non-zero coordinate, which finishes the proof.

The result of Barthe and Naor [BN] on the extremal hyperplane projections of the balls  $B_p^n$  with  $p \geq 2$  is as follows:

**Theorem 3.** [BN] *Let  $p \geq 2$  and  $\theta \in S^{n-1}$ , then*

$$\text{Vol}_{n-1}\left(B_p^n \Big|_{\theta_1^\perp}\right) \leq \text{Vol}_{n-1}\left(B_p^n \Big|_{\theta^\perp}\right) \leq \text{Vol}_{n-1}\left(B_p^n \Big|_{\theta_n^\perp}\right),$$

where  $\theta_1 = (1, 0, \dots, 0)$  and  $\theta_n = (1/\sqrt{n}, \dots, 1/\sqrt{n})$ .

The proof in [BN] is based on a formula similar to (7): for every  $\xi \in S^{n-1}$ ,

$$\text{Vol}_{n-1}\left(B_p^n \Big|_{\xi^\perp}\right) = \frac{2^n \Gamma^n(1/p)}{\pi p^{n-1} \Gamma(n - \frac{n-1}{p^*})} \int_0^\infty \frac{1 - \prod_{k=1}^n \frac{p^*}{2\Gamma(1/p)} \beta_{p^*}(t\xi_k)}{t^2} dt, \quad (8)$$

where  $1/p + 1/p^* = 1$  and  $\beta_{p^*}(u)$  is the Fourier transform of the function  $|x|^{p^*-2} e^{-|x|^{p^*}}$  on  $\mathbb{R}$ . The proof of this formula in [BN] uses probabilistic arguments. The rest of the proof of Theorem 3 is similar to that for sections. For  $p \geq 2$ , the function  $\beta_{p^*}(\sqrt{\cdot})$  is log-convex on  $[0, \infty)$ , which is proved in a way similar to the case of the function  $\gamma_p$  in [K5]. Theorem 3 immediately follows from this and (8).

In this section we show that formula (8) is a particular case of the formula (2) and holds for every  $p > 1$ . In particular, this allows us

to unify methods in the section and projection cases. We do that by computing the Fourier transform of the curvature function of  $l_p$ -balls using techniques from differential geometry (see [Th]) and a trick for calculating the Fourier transform from [K5].

**Lemma 1.** *Let  $B_p^n = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1\}$ ,  $1 < p < \infty$ . Then*

$$f_{B_p^n}(\theta) = (p^* - 1)^{n-1} \left( \prod_{i=1}^n |\theta_i|^{p^*-2} \right) \|\theta\|_{p^*}^{(n-1)-np^*}, \quad \theta \in S^{n-1}.$$

**Proof:** To find  $f_{B_p^n}(\theta)$  we first compute the Gaussian curvature  $K_p(x)$  of  $B_p^n$ . Let  $F(x_1, \dots, x_n) = \sum_{i=1}^n |x_i|^p$ , then the boundary  $\partial B_p^n = \{x \in \mathbb{R}^n : F(x_1, \dots, x_n) = 1\}$ . Let

$$Z(x) = -\frac{1}{p} \nabla F(x) = -(|x_1|^{p-1} \text{sign } x_1, \dots, |x_n|^{p-1} \text{sign } x_n).$$

We use the following formula for the Gaussian curvature (see, for example, [Th], Theorem 5, page 89):

$$K(x) = (-1)^{n-1} \det \begin{pmatrix} \nabla_{v_1} Z \\ \vdots \\ \nabla_{v_{n-1}} Z \\ Z(x) \end{pmatrix} / |Z(x)|^{n-1} \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ Z(x) \end{pmatrix}. \quad (9)$$

Here vectors  $v_1, \dots, v_{n-1}$  can be chosen as any basis in the tangent space at the corresponding point, and

$$\nabla_{v_i} Z = (\nabla_{v_i} Z_1, \dots, \nabla_{v_i} Z_n) = (v_i \cdot \nabla Z_1, \dots, v_i \cdot \nabla Z_n).$$

We choose  $v_1, \dots, v_{n-1}$  in the following way: for every  $k$ ,  $1 \leq k \leq n-1$ ,

$$v_k = (|x_{k+1}|^{p-1} \text{sign } x_{k+1}, \underbrace{0, \dots, 0}_{k-1 \text{ times}}, -|x_1|^{p-1} \text{sign } x_1, 0, \dots, 0).$$

Then

$$\nabla_{v_k} Z(x) = (p-1)|x_1 x_{k+1}|^{p-2} (-x_{k+1}, 0, \dots, 0, \underbrace{x_1}_{(k+1)\text{s place}}, 0, \dots, 0).$$

Substituting this in (9) (computation of the determinants is standard), we get

$$K_p(x) = (p-1)^{n-1} |Z(x)|^{-n-1} \prod_{i=1}^n |x_i|^{p-2} \quad (10)$$

for almost all  $x \in \partial B_p^n$ .

Let  $\theta = Z(x)/|Z(x)|$ . Recall that  $f_K(\cdot)$  is the reciprocal Gauss curvature, viewed as a function of the unit normal vector (see [Sc2], page 419), so  $f_{B_p^n}(\theta) = 1/K_p(Z(x)/|Z(x)|)$ .

Since  $\theta = Z(x)/|Z(x)|$ , we see that  $\theta$  and  $Z(x)$  are collinear vectors in  $\mathbb{R}^n$ , i.e.  $\theta = \lambda Z(x)$  for some  $\lambda > 0$ . Next

$$x_i = \lambda^{1/(p-1)} |\theta_i|^{1/(p-1)} \text{sign } \theta_i,$$

and using  $\|x\|_p = 1$  we get

$$\lambda = \frac{1}{\left(\sum_{i=1}^n |\theta_i|^{p/(p-1)}\right)^{(p-1)/p}}.$$

Finally

$$x_i = -(|\theta_i|^{1/(p-1)} \text{sign } \theta_i) / \left(\sum_{i=1}^n |\theta_i|^{p/(p-1)}\right)^{1/p}.$$

To finish the proof, we substitute the last expression into (10) and use  $p-1 = (p^* - 1)^{-1}$ .

□

**Theorem 4.** *The formula (8) holds for every  $p > 1$ .*

**Proof :** According to Theorem 2, we have to compute the Fourier transform of  $f_{B_p^n}(x) = |x|^{-n-1} f_{B_p^n}(x/|x|)$ ,  $x \in \mathbb{R}^n \setminus \{0\}$ . From the definition of the Gamma function

$$\|x\|_{p^*}^{-q} = \frac{p^*}{\Gamma(q/p^*)} \int_0^\infty y^{q-1} e^{-y^{p^*} (|x_1|^{p^*} + \dots + |x_n|^{p^*})} dy \quad (11)$$

for any  $q > 0$  and  $x \in \mathbb{R}^n \setminus \{0\}$ .

Let  $\phi \in \mathcal{S}(\mathbb{R}^n)$  be even and such that  $0 \notin \text{supp}(\widehat{\phi})$ . Using (11), we get

$$\begin{aligned} \langle \widehat{f_{B_p^n}}, \phi \rangle &= \langle f_{B_p^n}, \widehat{\phi} \rangle = \\ &= \frac{(p^* - 1)^{n-1} p^*}{\Gamma(q/p^*)} \int_0^\infty y^{q-1} dy \int_{\mathbb{R}^n} \prod_{k=1}^n |x_k|^{p^*-2} e^{-y^{p^*} |x_k|^{p^*}} \widehat{\phi}(x) dx = \\ &= \frac{(p^* - 1)^{n-1} p^*}{\Gamma(q/p^*)} \int_0^\infty y^{q-1} dy \int_{\mathbb{R}^n} \left( \prod_{k=1}^n |x_k|^{p^*-2} e^{-y^{p^*} |x_k|^{p^*}} \right)^\wedge (\xi) \phi(\xi) d\xi = \\ &= \frac{(p^* - 1)^{n-1} p^*}{\Gamma(q/p^*)} \int_0^\infty y^{q-1} dy \int_{\mathbb{R}^n} \prod_{k=1}^n \left( |t|^{p^*-2} e^{-y^{p^*} |t|^{p^*}} \right)^\wedge (\xi_k) \phi(\xi) d\xi. \end{aligned}$$

This gives

$$\langle \widehat{f_{B_p^n}}, \phi \rangle = \frac{(p^* - 1)^{n-1} p^*}{\Gamma(q/p^*)} \int_0^\infty y^{q-1+(1-p^*)n} dy \int_{\mathbb{R}^n} \prod_{k=1}^n \beta_{p^*} \left( \frac{\xi_k}{y} \right) \phi(\xi) d\xi.$$

Now replacing  $q$  by  $np^* - (n - 1)$ , and using the fact that  $0 \notin \text{supp}(\hat{\phi})$  implies  $\int_{\mathbb{R}^n} \phi(\xi) d\xi = 0$ , we obtain

$$\begin{aligned} \langle \widehat{f_{B_p^n}}, \phi \rangle &= \frac{(p^* - 1)^{n-1} p^*}{\Gamma(n - (n - 1)/p^*)} \int_0^\infty dy \int_{\mathbb{R}^n} \left( \prod_{k=1}^n \beta_{p^*} \left( \frac{\xi_k}{y} \right) - \beta_{p^*}^n(0) \right) \phi(\xi) d\xi = \\ &= \frac{(p^* - 1)^{n-1} p^*}{\Gamma(n - (n - 1)/p^*)} \int_{\mathbb{R}^n} \phi(\xi) d\xi \int_0^\infty \prod_{k=1}^n \beta_{p^*} \left( \frac{\xi_k}{y} \right) - \beta_{p^*}^n(0) dy, \end{aligned}$$

where

$$\beta_{p^*}(0) = \int_{-\infty}^\infty |u|^{p^*-2} e^{-|u|^{p^*}} du = \frac{2}{p^*} \int_0^\infty v^{-1/p^*} e^{-v} dv = \frac{2}{p^*} \Gamma(1 - 1/p^*).$$

Making a substitution  $\xi_k/y = t$ , we finally get

$$\begin{aligned} \langle \widehat{f_{B_p^n}}, \phi \rangle &= \frac{(p^* - 1)^{n-1} p^*}{\Gamma(n - (n - 1)/p^*)} \int_{\mathbb{R}^n} \phi(\xi) d\xi \int_0^\infty \frac{\prod_{k=1}^n \beta_{p^*}(t\xi_k) - \beta_{p^*}^n(0)}{t^2} dt = \\ &= \frac{2^n \Gamma^n(1/p)}{p^{n-1} \Gamma(n - (n - 1)/p^*)} \int_{\mathbb{R}^n} \phi(\xi) d\xi \int_0^\infty \frac{\prod_{k=1}^n \frac{p^*}{2\Gamma(1/p)} \beta_{p^*}(t\xi_k) - 1}{t^2} dt. \end{aligned}$$

Since  $\phi$  is an arbitrary even test function with  $0 \notin \text{supp}(\hat{\phi})$ , the distributions in the left and right-hand sides of the latter equality are equal (they are both even homogeneous of degree 1). Now the result follows from Theorem 2. □

#### 4. PROJECTION BODIES

Let  $K$  be an origin symmetric convex body in  $\mathbb{R}^n$ . The *projection body*  $\Pi K$  of  $K$  is defined as an origin symmetric convex body in  $\mathbb{R}^n$

whose support function in every direction is equal to the volume of the hyperplane projection of  $K$  in this direction: for every  $\theta \in S^{n-1}$ ,

$$h_{\Pi K}(\theta) = \text{Vol}_{n-1}(K|\theta^\perp) = \frac{1}{2} \int_{S^{n-1}} |\theta \cdot v| dS_{n-1}(K, v). \quad (12)$$

We extend  $h_{\Pi K}$  to a homogeneous function of degree 1 on  $\mathbb{R}^n$ . It immediately follows from Theorem 1 that the Fourier transform of  $h_{\Pi K}$  is equal (up to a constant) to the extended surface area measure of  $K$ :

$$\widehat{h_{\Pi K}} = -(2\pi)^{n-1} S_e(K). \quad (13)$$

On the other hand, if  $L$  is an origin symmetric convex body so that  $\widehat{h_L} = -(2\pi)^{n-1} \mu_e$  is an extended measure of some even finite Borel measure  $\mu$  on  $S^{n-1}$  then, again by Theorem 1, one has

$$h_L(\theta) = \frac{1}{2} \int_{S^{n-1}} |\theta \cdot v| d\mu(v).$$

Since  $L$  is an  $n$ -dimensional body, the measure  $\mu$  cannot be supported in any great subsphere of  $S^{n-1}$ , and, by Minkowski's existence theorem ([G], page 356), the measure  $\mu$  is the surface area measure of some origin symmetric convex body  $K$ . By Cauchy's formula, we have  $L = \Pi K$ . We have proved the following Fourier analytic characterization of projection bodies (this fact can also be shown as a combination of several well-known facts about zonoids and subspaces of  $L_1$ ):

**Proposition 1.** *An origin symmetric convex body  $L$  in  $\mathbb{R}^n$  is the projection body of some origin symmetric convex body if and only if there exists a measure  $\mu$  on  $S^{n-1}$  so that*

$$\widehat{h_L} = -(2\pi)^{n-1} \mu_e. \quad (14)$$

*The measure  $\mu$  serves as the surface area measure of an origin symmetric convex body  $K$  so that  $L = \Pi K$ .*

The condition that  $L$  is a projection body is equivalent to  $L$  being a centered zonoid (see [G], p. 134), which, in turn, is equivalent to the polar body  $L^*$  being the unit ball of a subspace of  $L_1$ . It is well-known that every origin symmetric convex body in  $\mathbb{R}^2$  is a projection body (or it is the unit ball of a subspace of  $L_1$ , see [He], [Fe], [Li]). As proved in [K1], for  $L = B_p^n$ ,  $0 < p < 2$ ,  $n \geq 3$  the function  $h_{B_p^n} = \|\cdot\|_{p^*}$  has a sign-changing Fourier transform, which implies that the spaces  $l_q^n$ ,  $q > 2$ ,  $n \geq 3$  do not embed isometrically in  $L_1$  (this fact was first proved by other methods by Dor [D]).

To get a characterization of projection bodies in terms of sections of the polar body, we use a formula from [GKS]. Let  $D$  be an infinitely

smooth origin symmetric convex body in  $\mathbb{R}^n$ ,  $k \in \mathbb{N} \cup \{0\}$ ,  $k \neq n-1$ ,  $\xi \in S^{n-1}$ . We denote by

$$A_{D,\xi}(t) = \text{Vol}_{n-1}(D \cap \{\xi^\perp + t\xi\}), \quad t \in \mathbb{R}$$

the parallel section function of  $D$  in the direction of  $\xi$ . Then:

(i) If  $k$  is even

$$(\|\cdot\|_D^{-n+k+1})^\wedge(\xi) = (-1)^{k/2} \pi(n-k-1) A_{D,\xi}^{(k)}(0); \quad (15)$$

(ii) If  $k$  is odd

$$(\|\cdot\|_D^{-n+k+1})^\wedge(\xi) = \quad (16)$$

$$c_{n,k} \int_0^\infty \frac{A_{D,\xi}(z) - A_{D,\xi}(0) - \dots - A_{D,\xi}^{(k-1)}(0) \frac{z^{k-1}}{(k-1)!}}{z^{k+1}} dz,$$

where  $c_{n,k} = (-1)^{(k+1)/2} 2(n-k-1)k!$ ,  $A_{D,\xi}^{(k)}$  stands for the derivative of the order  $k$  of the function  $A_{D,\xi}$ , and, as before, the equality of distributions on the sphere is understood as the equality of their homogeneous extensions.

Suppose now that an origin symmetric convex body  $L$  in  $\mathbb{R}^n$  has the property that  $h_L$  is an infinitely differentiable function on the sphere  $S^{n-1}$ . This means that the polar body  $L^*$  is infinitely smooth (recall that  $h_L = \|\cdot\|_{L^*}$ ). Putting  $k = n$ ,  $D = L^*$  in the formulas above, we get that if  $n$  is even then, for every  $\xi \in S^{n-1}$ ,

$$\widehat{h}_L(\xi) = (-1)^{1+n/2} \pi A_{L^*,\xi}^{(n)}(0) \quad (17)$$

and if  $n$  is odd then, for every  $\xi \in S^{n-1}$ ,

$$\widehat{h}_L(\xi) = \quad (18)$$

$$(-1)^{(n-1)/2} 2n! \int_0^\infty \frac{A_{L^*,\xi}(z) - A_{L^*,\xi}(0) - \dots - A_{L^*,\xi}^{(n-1)}(0) \frac{z^{n-1}}{(n-1)!}}{z^{n+1}} dz.$$

We can now characterize projection bodies in terms of sections of the polar body.

**Proposition 2.** *Let  $L$  be an origin symmetric convex body in  $\mathbb{R}^n$  so that  $h_L$  is infinitely differentiable on  $S^{n-1}$ . The body  $L$  is a projection body of some convex body  $K$  if and only if for every  $\xi \in S^{n-1}$ ,*

(i) if  $n$  is even

$$(-1)^{n/2} A_{L^*,\xi}^{(n)}(0) \geq 0;$$

(ii) if  $n$  is odd

$$(-1)^{(n+1)/2} \int_0^\infty \frac{A_{L^*,\xi}(z) - A_{L^*,\xi}(0) - \dots - A_{L^*,\xi}^{(n-1)}(0) \frac{z^{n-1}}{(n-1)!}}{z^{n+1}} dz \geq 0.$$

Using formulas (17), (18), one can express the volume of a convex body in terms of the  $(n - 1)$ -dimensional volumes of its hyperplane projections. If  $L^*$  is an infinitely smooth body and  $n$  is even then

$$\text{Vol}_n(L) = (-1)^{n/2} \frac{\pi^2}{(2\pi)^n} \int_{S^{n-1}} A_{L^*, \xi}^{(n)}(0) \text{Vol}_n(L|\xi^\perp) d\xi. \quad (19)$$

This formula is an analog of the formula expressing the volume of a body in terms of volumes of central hyperplane sections (see the remark before Theorem 4.5 in [K4]).

To prove (19) we recall that (see [G], page 354, [Sc2], page 275)

$$\text{Vol}_n(L) = \frac{1}{n} \int_{S^{n-1}} h_L(\theta) f_L(\theta) d\xi. \quad (20)$$

Using Parseval's formula on the sphere (see Appendix 1, Lemma 4, at the end of the paper) we have

$$\text{Vol}_n(L) = \frac{1}{(2\pi)^n} \int_{S^{n-1}} \widehat{h}_L(\xi) \widehat{f}_L(\xi) d\xi,$$

which, together with Theorem 2 and (17), gives (19).

## 5. A FOURIER ANALYTIC SOLUTION TO THE SHEPHARD PROBLEM

Let  $h_L(x) = \max\{x \cdot y : y \in L\}$  be the support function of a convex body  $L$ . By an approximation argument (see [Sc2], pages 158-160), we may assume in the formulation of Shephard's problem that the bodies  $K$  and  $L$  are such that  $h_K$ ,  $h_L$  are infinitely smooth functions on  $\mathbb{R}^n \setminus \{0\}$ . Using, for example, formulas (15), (16), we get in this case that the Fourier transforms  $\widehat{h}_K$ ,  $\widehat{h}_L$  are the extensions of infinitely differentiable functions on the sphere to homogeneous distributions on  $\mathbb{R}^n$  of degree  $-n - 1$ . Moreover, by the same approximation argument, we may assume that our bodies have absolutely continuous surface area measures. Therefore, in the rest of this paper,  $K$  and  $L$  are convex symmetric bodies with infinitely smooth support functions and absolutely continuous surface area measures.

### Theorem 5.

(i) If a body  $L$  is such that  $\widehat{h}_L \leq 0$  for all  $\theta \in S^{n-1}$  then the Shephard problem has an affirmative answer for this  $L$  and any  $K$ .

(ii) If  $f_K > 0$  for every  $\theta \in S^{n-1}$  and  $\widehat{h}_K$  is positive on an open subset of  $S^{n-1}$  then there exists a body  $L$  giving together with  $K$  a counterexample in Shephard's problem.

**Proof :** This theorem will follow from the next two lemmas and the fact that the condition

$$\text{Vol}_{n-1} \left( K \Big| \theta^\perp \right) \leq \text{Vol}_{n-1} \left( L \Big| \theta^\perp \right)$$

is equivalent (see Theorem 2) to

$$\widehat{f}_K(\theta) \geq \widehat{f}_L(\theta), \quad \theta \in S^{n-1}.$$

**Lemma 2.** *If  $\widehat{h}_L(\theta) \leq 0$ , and  $\widehat{f}_K(\theta) \geq \widehat{f}_L(\theta)$ ,  $\forall \theta \in S^{n-1}$ , then*

$$\text{Vol}_n(K) \leq \text{Vol}_n(L).$$

**Proof :** From  $\widehat{f}_K(\theta) \geq \widehat{f}_L(\theta)$  and  $\widehat{h}_L(\theta) \leq 0$  we get

$$\int_{S^{n-1}} \widehat{h}_L(\theta) \widehat{f}_K(\theta) d\theta \leq \int_{S^{n-1}} \widehat{h}_L(\theta) \widehat{f}_L(\theta) d\theta = (*).$$

Using Parseval's formula on the sphere (see Appendix 1 at the end of the paper) and (20),

$$(*) = (2\pi)^n \int_{S^{n-1}} h_L(\theta) f_L(\theta) d\theta = n(2\pi)^n \text{Vol}_n(L).$$

On the other hand,

$$\int_{S^{n-1}} \widehat{h}_L(\theta) \widehat{f}_K(\theta) d\theta = (2\pi)^n \int_{S^{n-1}} h_L(\theta) f_K(\theta) d\theta = n(2\pi)^n V_1(K, L),$$

where  $V_1(K, L)$  is the mixed volume (see [G], page 354, [Sc2], page 275). Thus,

$$V_1(K, L) \leq \text{Vol}_n(L).$$

Now we apply the Minkowski inequality ([Sc2], page 317):

$$V_1(K, L) \geq \text{Vol}_n(L)^{\frac{1}{n}} \text{Vol}_n(K)^{\frac{n-1}{n}}$$

to get

$$\text{Vol}_n(L) \geq \text{Vol}_n(L)^{\frac{1}{n}} \text{Vol}_n(K)^{\frac{n-1}{n}},$$

or

$$\text{Vol}_n(L) \geq \text{Vol}_n(K).$$

□

**Lemma 3.** *Let  $K$  be such that  $f_K(\theta) > 0 \forall \theta \in S^{n-1}$ . If  $\widehat{h}_K$  is positive on an open subset of  $S^{n-1}$ , then there exists a convex symmetric body  $L$  in  $\mathbb{R}^n$ , such that*

$$\widehat{f}_K \geq \widehat{f}_L,$$

but

$$\text{Vol}_n(K) > \text{Vol}_n(L).$$

**Proof :** Let  $\Omega = \{\theta \in S^{n-1} : \widehat{h}_K(\theta) > 0\}$  and let  $v \in C^\infty(S^{n-1})$  be a non-positive even function supported on  $\Omega$ ,  $v$  is not identically zero. We extend  $v$  to a homogeneous function  $rv(\theta)$  of degree 1 on  $\mathbb{R}^n$ . Then the Fourier transform of  $rv(\theta)$  is a homogeneous function of degree  $-n-1$ :  $\widehat{rv}(\theta) = r^{-n-1}g(\theta)$ , where  $g$  is an infinitely smooth function on  $S^{n-1}$  (see Lemma 5 from [K3]).

Since  $g$  is bounded on  $S^{n-1}$ , one can choose a small  $\varepsilon > 0$  so that, for every  $\theta \in S^{n-1}$  and  $r > 0$ ,

$$f_L(r\theta) = f_K(r\theta) + \varepsilon r^{-n-1}g(\theta) > 0.$$

By Minkowski's existence theorem (see [G], page 356, [Sc2], page 389),  $f_L(\theta)$  defines a convex symmetric body  $L \in \mathbb{R}^n$ . By the definition of the function  $v$ ,

$$\widehat{f}_L(r\theta) = \widehat{f}_K(r\theta) + \varepsilon rv(\theta) \leq \widehat{f}_K(r\theta).$$

Next, since  $v$  is supported and is non-positive in the set where  $\widehat{h}_K > 0$ ,

$$\begin{aligned} \int_{S^{n-1}} \widehat{h}_K(\theta) \widehat{f}_L(\theta) d\theta &= \int_{S^{n-1}} \widehat{h}_K(\theta) \widehat{f}_K(\theta) d\theta + \int_{S^{n-1}} \widehat{h}_K(\theta) \varepsilon v(\theta) d\theta < \\ &< \int_{S^{n-1}} \widehat{h}_K(\theta) \widehat{f}_K(\theta) d\theta = (2\pi)^n \int_{S^{n-1}} h_K(\theta) f_K(\theta) d\theta = n(2\pi)^n \text{Vol}_n(K). \end{aligned}$$

But again by Parseval's formula,

$$\int_{S^{n-1}} \widehat{h}_K(\theta) \widehat{f}_L(\theta) d\theta = (2\pi)^n \int_{S^{n-1}} h_K(\theta) f_L(\theta) d\theta = n(2\pi)^n V_1(L, K),$$

so

$$V_1(L, K) < \text{Vol}_n(K).$$

As in the previous lemma, this implies

$$\text{Vol}_n(K) > \text{Vol}_n(L).$$

□

**Theorem 6.** *The Shephard problem has an affirmative answer in  $\mathbb{R}^2$  and negative answer in  $\mathbb{R}^n$ ,  $n \geq 3$ .*

**Proof :** Let  $n = 2$ . In view of Theorem 5, it is enough to show that for any convex symmetric body  $L$  in  $\mathbb{R}^2$  such that  $h_L$  is infinitely smooth, we have  $\widehat{h}_L \leq 0$ . This follows from the fact that every two-dimensional normed space embeds in  $L_1$ , as mentioned in the previous section. We, however, prefer to give a direct Fourier analytic argument, similar to

that in the solution of the Busemann-Petty problem: by formula (17) with  $n = k = 2$ , for every  $\xi \in S^{n-1}$ ,

$$\widehat{h}_L(\xi) = \pi A_{L^*,\xi}''(0) \leq 0,$$

where the last inequality holds because the central section has maximal volume among all hyperplane sections perpendicular to a fixed direction, so the function  $A_{L^*,\xi}$  has maximum at zero.

Let  $n \geq 3$ . To apply Theorem 5, we need to give an example of a symmetric convex body  $K$  so that  $h_K$  is infinitely differentiable, positive on an open subset of  $S^{n-1}$  and  $f_K > 0$  on  $S^{n-1}$ . Let  $K'$  be the unit ball of  $\ell_{4/3}^n$ . Then the function  $h_{K'}(x) = \|x\|_{\ell_4^n}$  is infinitely smooth and  $\widehat{h}_{K'}$  is positive on some open subset of  $S^{n-1}$  (see [K1], §4; for the sake of completeness, we include a sketch of the proof in Appendix 2).

Define the body  $K : h_K = h_{K'} + \varepsilon h_{B_2^n}$ , where  $\varepsilon > 0$  is such that  $\widehat{h}_K$  is still positive on an open subset of  $S^{n-1}$ . Then  $f_K(\theta) > 0, \forall \theta \in S^{n-1}$ , and the result follows from Theorem 5. □

**Remarks.** (i) The result of Theorem 5 can be translated into the language of geometry using the Fourier transform characterization of projection bodies, see Proposition 1. What we get are the results of Petty and Schneider connecting Shephard's problem with projection bodies.

(ii) To explain why the transition in Shephard's problem occurs between the dimensions 2 and 3, let us look again at formulas (17) and (18). The sign of  $\widehat{h}_L$  basically depends on the sign of the derivative  $A_{L^*,\xi}^{(n)}(0)$ . If  $n = 2$  we can control this sign, thanks to the convexity of  $L^*$ . However, when  $n \geq 3$  convexity does not control the derivative of order  $n$ , which allows to construct counterexamples. Another way to do that is to follow (with very minor changes; just replace the exponent  $1/4$  in the definition of the function  $f_\varepsilon$  by  $1/2$ ) the counterexample to the Busemann-Petty problem from [GKS], Theorem 4.

APPENDIX 1: A VERSION OF PARSEVAL'S FORMULA ON THE SPHERE

**Lemma 4.**

$$\int_{S^{n-1}} \widehat{h}_L(\theta) \widehat{f}_K(\theta) d\theta = (2\pi)^n \int_{S^{n-1}} h_L(\theta) f_K(\theta) d\theta.$$

**Proof :** We follow [K3], Lemmas 2, 3.

**Proposition 3.** *Let  $E(t) = t^4 e^{-t^4}$ ,  $\mu_L(\xi) = \widehat{E(h_L)}(\xi)$ . Then for almost all (with respect to Lebesgue measure)  $\theta \in S^{n-1}$ ,*

$$\int_0^\infty t^n |\mu_L(t\theta)| dt < \infty.$$

**Proof :** Since  $\mu_L$  is a bounded function ( $h_L$  is homogeneous of degree 1, so  $E(h_L) \in L_1(\mathbb{R}^n)$ ),

$$\int_0^1 t^n |\mu_L(t\theta)| dt < \infty \quad \forall \theta \in S^{n-1}.$$

It remains to show that for almost all  $\theta$

$$\int_1^\infty t^n |\mu_L(t\theta)| dt < \infty. \quad (21)$$

But for an even  $n$  one may see that

$$\Delta^{\frac{n+2}{2}} [h_L^4 e^{-h_L^4}] \in L_1(\mathbb{R}^n). \quad (22)$$

In fact, after differentiation, the function in front of the exponent is the sum of homogeneous functions of degrees greater than  $-n$ , and each of them is continuous on the unit sphere.

Now, (22) implies that  $\xi \mapsto |\xi|^{n+2} \mu_L(\xi)$ ,  $\xi \in \mathbb{R}^n$  is a bounded function, since it is the Fourier transform of an  $L_1$ -function, hence

$$\int_{|\xi|>1} |\xi| |\mu_L(\xi)| d\xi < \infty.$$

Passing to the polar coordinates we get (21).

Finally, one can put  $\frac{n+1}{2}$  in place of  $\frac{n+2}{2}$  to prove result for odd  $n$ . □

**Proposition 4.** *Let  $\mu_L$  be as above and  $\phi \in \mathcal{S}$  with  $0 \notin \text{supp}(\phi)$ . Then*

$$\int_{\mathbb{R}^n} \mu_L(x) dx \int_0^\infty \phi(rx) \frac{dr}{r^2} = \frac{\Gamma(3/4)}{4} \int_{S^{n-1}} \widehat{h_L}(\theta) d\theta \int_0^\infty \phi(r\theta) \frac{dr}{r^2}. \quad (23)$$

**Proof :** Observe that  $\widehat{E(h_L)}(rx)(\xi) = r^{-n} \mu_L(\xi/r)$  for every  $r > 0$ . We have

$$\int_{\mathbb{R}^n} \mu_L(x) dx \int_0^\infty r^{-2} \phi(rx) dr = \int_0^\infty r^{-2} dr \int_{\mathbb{R}^n} \phi(rx) \mu_L(x) dx =$$

$$\begin{aligned} \int_0^\infty r^{-2} dr \int_{\mathbb{R}^n} \phi(\xi) \mu_L(\xi/r) r^{-n} d\xi &= \int_0^\infty r^{-2} dr \int_{\mathbb{R}^n} \widehat{\phi}(y) E(r h_L(y)) dy = \\ &= \int_{\mathbb{R}^n} \widehat{\phi}(y) dy \int_0^\infty r^{-2} E(r h_L(y)) dr. \end{aligned}$$

By making a substitution  $r h_L(y) = t$  in the last integral and using the fact that

$$\int_0^\infty t^2 e^{-t^4} dt = \frac{\Gamma(3/4)}{4},$$

we get the desired result

$$\begin{aligned} \int_{\mathbb{R}^n} \mu_L(x) dx \int_0^\infty r^{-2} \phi(rx) dr &= \frac{\Gamma(3/4)}{4} \int_{\mathbb{R}^n} h_L(y) \widehat{\phi}(y) dy = \\ &= \frac{\Gamma(3/4)}{4} \langle h_L, \widehat{\phi} \rangle = \frac{\Gamma(3/4)}{4} \langle \widehat{h}_L, \phi \rangle = \\ &= \frac{\Gamma(3/4)}{4} \int_{\mathbb{R}^n} \widehat{h}_L(\xi) \phi(\xi) d\xi = \int_{S^{n-1}} \widehat{h}_L(\theta) d\theta \int_0^\infty r^{-2} \phi(r\theta) dr. \end{aligned}$$

□

**Proposition 5.** *Let  $\mu_L$  be as above. Then*

$$\int_0^\infty r^n \mu_L(r\theta) dr = \frac{\Gamma(3/4)}{4} \widehat{h}_L(\theta), \quad \forall \theta \in S^{n-1}. \quad (24)$$

**Proof :** Let  $\phi \in \mathcal{S}$  be such that  $0 \notin \text{supp}(\phi)$ . Passing to the polar coordinates, using Proposition 3 and (23), we get

$$\begin{aligned} \int_{\mathbb{R}^n} \mu_L(x) dx \int_0^\infty \phi(rx) \frac{dr}{r^2} &= \int_{S^{n-1}} \left( \int_0^\infty t^n \mu_L(t\theta) dt \right) \left( \int_0^\infty \phi(r\theta) \frac{dr}{r^2} \right) d\theta = \\ &= \frac{\Gamma(3/4)}{4} \int_{S^{n-1}} \widehat{h}_L(\theta) \left( \int_0^\infty \phi(r\theta) \frac{dr}{r^2} \right) d\theta. \end{aligned}$$

Now we put  $\phi(r\theta) = u(r)v(\theta)$ , where  $v$  is any infinitely smooth function on  $S^{n-1}$  and  $u$  is a non-negative test function on  $\mathbb{R}$ , such that  $0 \notin$

supp  $u$ . This gives:

$$\int_{S^{n-1}} \left( \int_0^\infty t^n \mu_L(t\theta) dt \right) v(\theta) d\theta = \frac{\Gamma(3/4)}{4} \int_{S^{n-1}} \widehat{h}_L(\theta) v(\theta) d\theta.$$

for every  $v \in C^\infty(S^{n-1})$ .

□

**Proposition 6.** *Under the same notation,*

$$\int_{\mathbb{R}^n} f_K(x) E(h_L)(x) dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}_K(\xi) \mu_L(\xi) d\xi.$$

**Proof :** First, note that both integrals in the latter formula converge absolutely, because  $h_L$  is a homogeneous function of degree 1 and by Proposition 3. Since  $E(h_L)(0) = 0$ , we have

$$\int_{\mathbb{R}^n} f_K(x) E(h_L)(x) dx = \int_{S^{n-1}} f_K(\theta) d\theta \int_0^\infty \frac{E(h_L)(r\theta) - E(h_L)(0)}{r^2} dr.$$

Let  $\gamma_\varepsilon$  be the standard Gaussian density with variance  $\varepsilon$ . Then the convolution  $E(h_L) * \gamma_\varepsilon$  is an *even* test function. Hence, the integral

$$\int_0^\infty \frac{(E(h_L) * \gamma_\varepsilon)(r\theta) - (E(h_L) * \gamma_\varepsilon)(0)}{r^2} dr$$

is well defined for any  $\varepsilon > 0$ , and is equal to  $1/2 \langle r^{-2}, (E(h_L) * \gamma_\varepsilon)(r\theta) \rangle$  (see [GS], page 52, formula (7)). Splitting it into two integrals over  $[0, 1]$  and  $[1, \infty)$  and using the fact that the derivatives up to order 4 of the function  $F(r) \equiv (E(h_L) * \gamma_\varepsilon)(r\theta)$  are uniformly (with respect to  $\varepsilon$ ) bounded, we get

$$\lim_{\varepsilon \rightarrow 0} \int_0^\infty \frac{(E(h_L) * \gamma_\varepsilon)(r\theta) - (E(h_L) * \gamma_\varepsilon)(0)}{r^2} dr = \int_0^\infty \frac{E(h_L)(r\theta)}{r^2} dr$$

This, together with the definition of extended measure (5) and the dominated convergence theorem gives:

$$\begin{aligned} \int_{\mathbb{R}^n} f_K(x) E(h_L)(x) dx &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{S^{n-1}} \langle r^{-2}, (E(h_L) * \gamma_\varepsilon)(r\theta) \rangle f_K(\theta) d\theta = \\ &= \lim_{\varepsilon \rightarrow 0} \langle f_K, E(h_L) * \gamma_\varepsilon \rangle = \lim_{\varepsilon \rightarrow 0} \frac{1}{(2\pi)^n} \langle \widehat{f}_K, (E(h_L) * \gamma_\varepsilon)^\wedge \rangle = \end{aligned}$$

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f_K}(\xi) \widehat{E(h_L)}(\xi) \widehat{\gamma_\varepsilon}(\xi) d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f_K}(\xi) \mu_L(\xi) d\xi.$$

□

**Proposition 7.** *Under the same notation,*

$$\frac{\Gamma(3/4)}{4} \int_{S^{n-1}} f_K(\theta) h_L(\theta) d\theta = \frac{1}{(2\pi)^n} \int_{S^{n-1}} \widehat{f_K}(\theta) \int_0^\infty r^n \mu_L(r\theta) dr d\theta.$$

**Proof :** Note that

$$\frac{\Gamma(3/4)}{4} h_L(\theta) = \int_0^\infty r^{-2} E(h_L)(r\theta) dr. \quad (25)$$

Passing to polar coordinates and using (25), Proposition 6, and the fact that  $f_K$  is a homogeneous function of degree  $-n - 1$  on  $\mathbb{R}^n$ , we get

$$\begin{aligned} \frac{\Gamma(3/4)}{4} \int_{S^{n-1}} f_K(\theta) h_L(\theta) d\theta &= \int_{\mathbb{R}^n} f_K(x) E(h_L)(x) dx = \\ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f_K}(\xi) \mu_L(\xi) d\xi &= \frac{1}{(2\pi)^n} \int_{S^{n-1}} \widehat{f_K}(\theta) d\theta \int_0^\infty r^n \mu_L(r\theta) dr. \end{aligned}$$

□

We finish the proof of Lemma 4 by comparing Propositions 5 and 7.

□

## APPENDIX 2: THE FOURIER TRANSFORM OF $\|\cdot\|_4$

First, using the same trick with the Gamma function, as in Theorem 4, we get that the Fourier transform of the distribution  $\|\cdot\|_4$  in  $\mathbb{R}^n$  is equal to

$$(\|\cdot\|_4)^\wedge(\xi) = \frac{4}{\Gamma(-1/4)} \int_0^\infty t^n \prod_{k=1}^n \gamma_4(t\xi_k) dt, \quad (26)$$

for every  $\xi \in \mathbb{R}^n$ ,  $\xi \neq 0$ , where  $\gamma_4$  is the Fourier transform of the function  $z \mapsto \exp(-z^4)$ ,  $z \in \mathbb{R}$ . Note that  $\gamma_4(t)$  decreases at infinity exponentially (as  $\exp(-|t|^{4/3})$ ), which explains why is the function in the right-hand side of the latter formula infinitely differentiable on  $S^{n-1}$  (this is no longer true for the Fourier transform of  $\|\cdot\|_q$  where  $q$  is not an even integer; in this case  $\|\cdot\|_q$  is not infinitely smooth on the sphere, and  $\gamma_q(t)$  decreases at infinity as  $|t|^{-1-q}$ ).

Secondly, the moments of the function  $\gamma_4$  converge absolutely and can be computed using Parseval's formula: for every  $-1 < \alpha < \infty$ ,

$$s_4(\alpha) = \int_{\mathbb{R}} |t|^\alpha \gamma_4(t) dt = \frac{2^\alpha \sqrt{\pi} \Gamma(-\alpha/4) \Gamma((\alpha+1)/2)}{\Gamma(-\alpha/2)}.$$

It is easily seen that the moments  $s_4(\alpha)$  are positive for  $\alpha \in (-1, 0) \cup (0, 2]$  and negative for  $\alpha \in (2, 4)$ .

For  $\alpha_1, \dots, \alpha_{n-1} \in (-1, \infty)$ , consider the integral

$$I(\alpha_1, \dots, \alpha_{n-1}) = \int_{\mathbb{R}^{n-1}} |\xi_1|^{\alpha_1} \dots |\xi_{n-1}|^{\alpha_{n-1}} (\|\cdot\|_4)^{\wedge}(\xi_1, \dots, \xi_{n-1}, 1) d\xi_1 \dots d\xi_{n-1}.$$

By (26), the latter integral is equal to

$$I(\alpha_1, \dots, \alpha_{n-1}) = \frac{4}{\Gamma(-1/4)} s_4(\alpha_1) \dots s_4(\alpha_{n-1}) s_4(-\alpha_1 - \dots - \alpha_{n-1} + 1).$$

Choose  $\alpha_1, \dots, \alpha_{n-1} \in (-1, 0)$  so that  $-\alpha_1 - \dots - \alpha_{n-1} \in (1, 2)$ , which is possible because  $n-1 \geq 2$ . Then the signs of the moments are such that  $I(\alpha_1, \dots, \alpha_{n-1})$  is positive, which means that the function  $\widehat{\|\cdot\|_4}$  cannot be negative everywhere. Since this function is even homogeneous on  $\mathbb{R}^n$  and continuous on the sphere, it is positive on some open subset of  $S^{n-1}$ .

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