FOURIER ANALYTIC METHODS IN THE STUDY OF PROJECTIONS AND SECTIONS OF CONVEX BODIES

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It has been noticed long ago that many results on sections and projections are dual to each other, though methods used in the proofs are quite different and don't use the duality of underlying structures directly. In the paper [KRZ], the authors attempted to start a unified approach connecting sections and projections, which may eventually explain these mysterious connections. The idea is to use the recently developed Fourier analytic approach to sections of convex bodies (a short description of this approach can be found in [K7]) as a prototype of a new approach to projections. The first results seem to be quite promising. The crucial role in the Fourier approach to sections belongs to certain formulas connecting the volume of sections with the Fourier transform of powers of the Minkowski functional. An analog of these formula for the case of projections was found in [KRZ] and connects the volume of projections to the Fourier transform of the curvature function. This formula was applied in [KRZ] to give a new proof of the result of Barthe and Naor on the extremal projections of l_p-balls with $p > 2$, which is similar to the proof of the result on the extremal sections of l_p-balls with $0 < p < 2$ in [K5]. Another application is to the Shephard problem, asking whether bodies with smaller hyperplane projections necessarily have smaller volume. The problem was solved independently by Petty and Schneider, and the answer is affirmative in the dimension two and negative in the dimensions three and higher. The paper [KRZ] gives a new Fourier analytic solution to this problem that essentially follows the Fourier analytic solution to the Busemann-Petty problem (the projection counterpart of Shephard's problem) from [K3]. The transition in the Busemann-Petty problem occurs between the dimensions four and five. In Section 4, we show that the transition in both problems has the same explanation based on similar Fourier analytic characterizations of intersection and projection bodies.

The goal of this survey is to bring together certain aspects of the Fourier approaches to sections and projections, in order to emphasize the similarities between the results and the proofs. We do not include the proofs and refer the reader to [K7] and [KRZ] for complete proofs, other related results and references.

1. Volume and the Fourier transform

We start with necessary notations and definitions. The Minkowski functional of a convex body K is defined by

$$||x||_K = \min\{a \geq 0 : x \in a\,K\}.$$

Observe that $K = \{x : ||x||_K \leq 1\}$. If χ is the indicator function of the interval $[-1,1]$, and $\xi \in S^{n-1}$ then

$$\mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \int\limits_{x\in\mathbb{R}^n:\, x\cdot\xi=0} \chi(||x||_K)dx.$$

Passing to polar coordinates in the hyperplane $x \cdot \xi = 0$, we get the *polar formula for the volume of sections*:

$$\mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{n-1}\int_{S^{n-1}\cap\xi^\perp} ||\theta||_K^{-n+1}d\theta. \tag{1}$$

It is important to note that the right-hand side of the above formula is the spherical Radon transform of $||\cdot||_K^{-n+1}$.

The surface area measure $S_{n-1}(K,\cdot)$ of a convex body K in $\mathbb{R}^n$ is defined as follows: for every Borel set $E \subset S^{n-1}$, $S_{n-1}(K,E)$ is equal to Lebesgue measure of the part of the boundary ∂K where normal vectors belong to E (see, for example [Ga3], page 351). The well-known Cauchy formula ([Ga3], page 361) expresses the volume of projections of the body K as the cosine transform of the surface area measure:

$$\mathrm{Vol}_{n-1}\left(K\middle|\theta^\perp\right) = \frac{1}{2}\int\limits_{S^{n-1}} |\theta\cdot v|dS_{n-1}(K,v), \qquad \theta\in S^{n-1}. \tag{2}$$

For our needs, it is enough to consider bodies with absolutely continuous surface area measures. A convex body K is said to have the curvature function

$$f_K(\cdot) : S^{n-1} \to \mathbb{R},$$

if its surface area measure $S_{n-1}(K,\cdot)$ is absolutely continuous with respect to Lebesgue measure σ_{n-1} on S^{n-1}, and

$$\frac{dS_{n-1}(K,\cdot)}{d\sigma_{n-1}} = f_K(\cdot) \in L_1(S^{n-1}).$$

We also recall that $f_K(\cdot)$ is the reciprocal Gauss curvature, viewed as a function of the unit normal vector (see [Sc2], page 419). In the next section, we apply this property to compute the volume of projections of l_p-balls.

By expressing the volume of sections and projections in terms of the spherical Radon and cosine transform one can reduce geometric problems to the study of these transforms. The Fourier analytic approach to sections and projections is based on relating these transforms to the Fourier transform first, and then applying methods of harmonic analysis to solve geometric problem. In many cases we operate with the Fourier transform of distributions. We denote by $\mathcal{S}$ the space of rapidly decreasing infinitely differentiable functions (test functions) on $\mathbb{R}^n$ with values in $\mathbb{C}$. By $\mathcal{S}'$ we denote the space of distributions over $\mathcal{S}$. Every locally integrable real valued function f on $\mathbb{R}^n$ with power growth at infinity represents a distribution acting by integration: for every $\phi \in \mathcal{S}$, $\langle f, \phi\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx$. The Fourier transform of a distribution f is defined by $\langle \hat{f}, \hat{\phi}\rangle = (2\pi)^n\langle f, \phi\rangle$, for every test function ϕ.

The following formula expressing the volume of hyperplane sections in terms of the Fourier transform was proved in [K5] by using a connection between the Radon and Fourier transforms of homogeneous distributions:

Let K be an origin symmetric star body in $\mathbb{R}^n$ and let $\xi \in S^{n-1}$. Then

$$\mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi(n-1)}(\|x\|_K^{-n+1})^\wedge(\xi). \tag{3}$$

An analog of this formula for the volume of projections was proved in [KRZ] by relating the cosine and Fourier transforms of homogeneous distributions:

Let K be a convex origin symmetric body in $\mathbb{R}^n$ with an absolutely continuous surface area measure. Then

$$\mathrm{Vol}_{n-1}\left(K\middle|\theta^\perp\right) = -\frac{1}{\pi}\widehat{f_K}(\theta), \qquad \forall\theta \in S^{n-1}. \tag{4}$$

Here $f_K(x) = |x|^{-n-1} f_K(x/|x|),\ x \in \mathbb{R}^n\setminus\{0\}$ is the extension of $f_K(x)$, $x \in S^{n-1}$ to a homogeneous function of degree $-n-1$. In the case of general convex bodies one has to use the so-called extended surface area measure $S_e(K)$ in place of the curvature function. An interested reader may find a brief description of this and other related concepts in the Appendix at the end of the paper.

2. Extremal sections and projections of B_p^n.

It has been known for a while that Fourier analytic formulas for the volume of sections are useful in the study of extremal sections of certain bodies. The first formula of this kind, relating the volume of sections of the unit cube $B^n_\infty = [-1,1]^n$ to the Fourier transform, was known to Polya [Po]:

$$\mathrm{Vol}(B_n^\infty \cap \xi^\perp) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\prod_{k=1}^n \frac{\sin(r\xi_k)}{r\xi_k}dr. \tag{5}$$

This formula has many applications, the most remarkable of them being the result of Ball [Ba1] that the maximal volume of hyperplane sections of the cube (in every dimension) is $\sqrt{2}$ and is attained at $\xi = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\ldots, 0)$. It is worth mentioning that finding the smallest hyperplane section of the cube is much easier and does not involve the Fourier transform (the minimal section is the one parallel to the face, as was first proved by Hadwiger [Ha]).

An analog of formula (5) for l_p-balls

$$B^n_p = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \le 1\}$$

was established by Meyer and Pajor [MP] for $1 \le p \le 2$ using probabilistic methods: for every $\xi \in S^{n-1}$

$$\mathrm{Vol}_{n-1}(B^n_p \cap \xi^\perp) = \frac{p}{\pi(n-1)\Gamma((n-1)/p)}\int_0^\infty \prod_{k=1}^n \gamma_p(t\xi_k)\ dt, \tag{6}$$

where γ_p is the Fourier transform of the function $\exp(-|\cdot|^p)$ on $\mathbb{R}$.. It was shown in [K5] that the latter formula works for all $p \in (0,\infty)$ and is a direct consequence of formula (3).

In particular, this formula allows to find the extremal sections of B^n_p when $0 < p \le 2$ (the upper bound for every $p \in [1,2]$ and the lower bound for $p = 1$ where found by Meyer and Pajor [MP], the lower bound for other values of p was proved in [K5]): for every $\theta \in S^{n-1}$,

$$\mathrm{Vol}_{n-1}\left(B^n_p \cap \theta_n^\perp\right) \le \mathrm{Vol}_{n-1}\left(B^n_p \cap \theta^\perp\right) \le \mathrm{Vol}_{n-1}\left(B^n_p \cap \theta_1^\perp\right),$$

where $\theta_1 = (1,0,\ldots,0)$ and $\theta_n = (1/\sqrt{n},\ldots,1/\sqrt{n})$. To deduce this result from (6), one only has to show that, for $0 < p \le 2$, the function $\gamma_p(\sqrt{\cdot})$ is log-convex on $[0,\infty)$, which was done in [K5]. In fact, log-convexity means that, for every $t > 0$ and $0 < \xi_1 < \eta_1 < \eta_2 < \xi_2$ with $\xi_1^2 + \xi_2^2 = \eta_1^2 + \eta_2^2 = 1$, one has

$$\gamma_p(t\xi_1)\gamma_p(t\xi_2) \ge \gamma_p(t\eta_1)\gamma_p(t\eta_2),$$

which immediately implies the result.

Now we pass the projection counterpart of the result for sections of l_p-balls. The case of hyperplane projections of the cube is quite simple.

Using the Cauchy formula (all the facets of the unit cube have the same volume, and their normal vectors are parallel to the axes), we get

$$\mathrm{Vol}_{n-1}\left(B_\infty^n \Big| \theta^\perp\right) = \mathrm{Vol}_{n-1}(B_\infty^{n-1}) \left(\sum_{i=1}^n |\theta_i|\right).$$

Then

$$\mathrm{Vol}\left(B_\infty^n \Big| (1,0,\dots,0)^\perp\right) \le \mathrm{Vol}\left(B_\infty^n \Big| \theta^\perp\right) \le \mathrm{Vol}\left(B_\infty^n \Big| (\frac{1}{\sqrt{n}},\cdots,\frac{1}{\sqrt{n}})^\perp\right),$$

or

$$\mathrm{Vol}_{n-1}(B_\infty^{n-1}) \le \mathrm{Vol}_{n-1}\left(B_\infty^n \Big| \theta^\perp\right) \le \sqrt{n}\mathrm{Vol}_{n-1}(B_\infty^{n-1}).$$

The same idea can be applied to the case of projections of B_1^n. Consider vectors $\varepsilon = (\varepsilon_1 \cdots \varepsilon_n)$, where ε_i can be chosen as $+1$ or -1. Then $\frac{1}{\sqrt{n}}\varepsilon$ are normal vectors to the facets of B_1^n. Let C_n be the volume of a facet of B_1^n, then again the Cauchy formula implies

$$\mathrm{Vol}_{n-1}\left(B_1^n \Big| \theta^\perp\right) = \frac{1}{\sqrt{n}} C_n \left(\sum_\varepsilon |\theta\cdot\varepsilon|\right).$$

Finding the maximal and minimal values of $\sum_\varepsilon |\theta\cdot\varepsilon|$ is related to sharp constants in the Khinchine inequality for independent symmetric Bernoulli random variables. These constants were found by Szarek [Sz]:

$$\mathrm{Vol}\left(B_1^n \Big| (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,\dots,0)^\perp\right) \le \mathrm{Vol}\left(B_1^n \Big| \theta^\perp\right) \le$$
$$\mathrm{Vol}\left(B_1^n \Big| (1,0,\dots,0)^\perp\right).$$

Comparing the results on extremal sections and projections of B_1^n and its polar B_∞^n, we see that the answers for projections are "dual" to the answer for sections.

Recently, Barthe and Naor [BN] have found extremal hyperplane projections of l_p-balls, $p > 2$. The result is "dual" to that for sections: for every $\theta \in S^{n-1}$,

$$\mathrm{Vol}_{n-1}\left(B_p^n \Big| \theta_1^\perp\right) \le \mathrm{Vol}_{n-1}\left(B_p^n \Big| \theta^\perp\right) \le \mathrm{Vol}_{n-1}\left(B_p^n \Big| \theta_n^\perp\right),$$

where $\theta_1 = (1,0,\dots,0)$ and $\theta_n = (1/\sqrt{n},\dots,1/\sqrt{n})$. The proof in [BN] is based on a formula similar to (6): for every $\xi \in S^{n-1}$,

$$\mathrm{Vol}_{n-1}\left(B_p^n \Big| \xi^\perp\right) = \frac{2^n\Gamma^n(1/p)}{\pi p^{n-1}\Gamma(n-\frac{n-1}{p^*})} \int_0^\infty \frac{1-\prod_{k=1}^n \frac{p^*}{2\Gamma(1/p)}\beta_{p^*}(t\xi_k)}{t^2}dt, \tag{7}$$

where $1/p+1/p^* = 1$ and $\beta_{p^*}(u)$ is the Fourier transform of the function $|x|^{p^*-2}e^{-|x|^{p^*}}$ on $\mathbb{R}$. The proof of this formula in [BN] uses probabilistic arguments. The rest of the proof in [BN] is similar to that for sections: for $p \geq 2$, the function $\beta_{p^*}(\sqrt{\cdot})$ is log-convex on $[0,\infty)$, which together with (7) immediately implies the result. It was shown in [KRZ] that formula (7) follows directly from (4) which makes the proof for projections completely similar ("dual") to that for sections.

In fact, f_K is the reciprocal Gauss curvature of K, viewed as a function of the unit normal vector. Thus, computing the Gauss curvature of B_p^n one gets (see [KRZ]) :

$$f_{B_p^n}(\theta) = (p^*-1)^{n-1}\left(\prod_{i=1}^n |\theta_i|^{p^*-2}\right)\|\theta\|_{p^*}^{(n-1)-np^*}, \qquad \theta \in S^{n-1},$$

and so (7) can be proved by computing the Fourier transform of $f_{B_p^n}$.

3. The Busemann-Petty and Shephard problems.

The Shephard problem (see [Sh]) reads as follows. Let K, L be convex symmetric bodies in $\mathbb{R}^n$ and suppose that, for every $\theta \in S^{n-1}$,

$$\mathrm{Vol}_{n-1}\left(K\Big|\theta^\perp\right) \leq \mathrm{Vol}_{n-1}\left(L\Big|\theta^\perp\right). \tag{8}$$

Does it follow that $\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)$? The problem was solved independently by Petty [P] and Schneider [Sc1], who showed that the answer is affirmative if $n \leq 2$ and negative if $n \geq 3$. Ball [Ba2], proved that it is necessary to multiply $\mathrm{Vol}_n(L)$ by $\sqrt{n}$ to make the answer affirmative in all dimensions. A version of the Shephard problem with lower dimensional projections was solved by Goodey and Zhang [GZ].

One of the main steps in the solution of the Shephard problem is a connection to projection bodies found by Schneider [Sc1]. Recall that an origin symmetric convex body L in $\mathbb{R}^n$ is called a projection body if there exists another convex body K so that the support function of L in every direction is equal to the volume of the hyperplane projection of K to this direction: for every $\theta \in S^{n-1}$, $h_L(\theta) = \mathrm{Vol}_{n-1}(K\Big|\theta^\perp)$. The support function $h_L(\theta) = \max_{x\in L}(\theta\cdot x)$ is equal to the dual norm $\|\cdot\|_{L^*}$, where L^* stands for the polar body of L.

Schneider [Sc1] discovered that if L is a projection body then the answer to the Shephard problem is affirmative for every K, and, on the other hand, if K is not a projection body one can perturb it to construct a body L giving together with K a counterexample. Therefore, the answer to the Shephard problem in $\mathbb{R}^n$ is affirmative if and only if every symmetric convex body in $\mathbb{R}^n$ is a projection body.

The section counterpart of Shephard's problem is the Busemann-Petty problem, posed in 1956 (see [BP]). Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that $\mathrm{Vol}_{n-1}(K \cap H) \leq \mathrm{Vol}_{n-1}(L \cap H)$ for every central hyperplane H in $\mathbb{R}^n$. Does it follow that $\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)$? The answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. The solution appeared as the result of a sequence of papers [LR], [Ba1], [Gi], [Bo], [Lu], [Pa], [Ga1], [Ga2], [Zh2], [K4], [K5], [Zh3], [GKS] (see [Zh3] and [GKS] for historical details).

The class of intersection bodies introduced by Lutwak [Lu] in 1988 plays the same role in the solution of the Busemann-Petty problem, as projection bodies in the solution to Shephard's problem. Let K and L be symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$, $\|\xi\|_K = \mathrm{Vol}_{n-1}(L \cap \xi^\perp)$. A more general class of *intersection bodies* can be defined as the closure of intersection bodies of star bodies in the radial metric $d(K, L) = \sup_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|$.

Lutwak [Lu] found the following connection between intersection bodies and the Busemann-Petty problem (the original result of Lutwak was slightly improved in [Ga1] and [Zh1]): if K is an intersection body then the answer to the BP-problem is affirmative for every L, and, on the other hand, if L is not an intersection body one can perturb it to construct a body K giving together with L a counterexample. Therefore, the answer to the Busemann-Petty problem in $\mathbb{R}^n$ is affirmative if and only if every symmetric convex body in $\mathbb{R}^n$ is an intersection body.

We would like to mention several facts concerning intersection and projection bodies, which can be found in [K6], [K4]:

The unit ball of any $n-$dimensional subspace of L^q with $0 < q < 2$ is an intersection body.

In particular the unit balls of ℓ_p^n for $0 < p < 2$ are intersection bodies.

For $2 < p \leq \infty$, the unit ball of the space ℓ_p^n is an intersection body if and only if $n \leq 4$.

As stated above, the unit ball of any subspace of L^1 is an intersection body. On the other hand, if K is a projection body, then K^* is the unit ball of a subspace of L_1 (Bolker, [Bl]). Thus:

The dual body of projection body is an intersection body.

Note that the converse is not true. Indeed, a cube in $\mathbb{R}^3$ is an intersection body, but the dual body, which is the cross-polytope, is not a projection body.

Let us outline the analytic solution to the Busemann-Petty problem from [GKS]. The first ingredient is a Fourier analytic characterization of intersection bodies from [K4]: an origin symmetric star body L in $\mathbb{R}^n$ is an intersection body if and only if the function $\|\cdot\|_L^{-1}$ represents a positive definite distribution on $\mathbb{R}^n$. Next, it was proved in [GKS] that if L is an infinitely smooth convex body in $\mathbb{R}^n$, and $k \neq n-1$ is an even integer, then the Fourier transform

$$(\|\cdot\|_L^{-n+k})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_{L,\xi}^{(k)}(0), \tag{9}$$

where

$$A_{L,\xi}(t) = \mathrm{Vol}_{n-1}(L \cap \{\xi^\perp + t\xi\}), \qquad t \in \mathbb{R}$$

is the parallel section function (or x-ray function) of L in the direction of ξ (if k is odd the right-hand side of (9) is a little more complicated). This formula, together with the Fourier analytic characterization of intersection bodies, shows that if n is an even integer then an infinitely smooth convex body $L \in \mathbb{R}^n$ is an intersection body if and only if

$$(-1)^{\frac{n-2}{2}} A_{L,\xi}^{(n-2)}(0) \geq 0, \qquad \forall \xi \in S^{n-1}. \tag{10}$$

If the body L is convex then, by Brunn's theorem, the central section is maximal among all sections orthogonal to a given direction, so $A''_{L,\xi}(0) \leq 0$ for every $\xi \in S^{n-1}$, and, therefore, putting $n = 4$ in (10) we see that every symmetric convex body in $\mathbb{R}^4$ is an intersection body. However, if $n = 5$ we have to deal with the third derivative of $A_{L,\xi}$ which is not controlled by convexity, and one can easily construct symmetric convex bodies in $\mathbb{R}^5$ that are not intersection bodies. This explains the transition between the dimensions 4 and 5 in the solution to the Busemann-Petty problem.

We now outline some ideas of the Fourier analytic proof of Shephard's problem in [KRZ]. First of all, projection bodies admit a Fourier analytic characterization similar to that for intersection bodies: an origin symmetric convex body $L \in \mathbb{R}^n$ is a projection body if and only if the Fourier transform of $\|\cdot\|_{L^*} = h_L(\cdot)$ is a negative distribution outside of the origin (this is essentially a combination of results of Bolker [Bl], who proved that projection bodies are polars of unit balls of subspaces of L_1, and P.Levy [Le], who characterized subspaces of L_1 in terms of negative definite functions). As pointed out in [KRZ], this Fourier characterization, together with formula (9) (put $k = n$), implies that if n is even then an origin symmetric convex body $L \in \mathbb{R}^n$ is a projection body if and only if

$$(-1)^{n/2} A_{L^*,\xi}^{(n)}(0) \geq 0, \qquad \forall \xi \in S^{n-1}.$$

Again, since convexity controls the derivatives only up to the second order, we conclude that every symmetric convex body in $\mathbb{R}^2$ is a projection body, but this is not the case in $\mathbb{R}^3$. This explains the transition between the dimensions 2 and 3 in the Shephard problem.

The result of Lutwak relating intersection bodies to the Busemann-Petty problem and the result of Schneider relating projection bodies to the Shephard problem admit almost identical proofs based on a spherical version of the Parseval formula (see [K3], [KRZ] for the cases of sections and projections, respectively).

Let us outline the Fourier analytic proofs of the "affirmative" parts of Lutwak's and Schneider's connections. First, Lutwak's result is that if K is an intersection body then the answer to the question of the Busemann-Petty problem is affirmative for this K and any symmetric convex body L. We translate this statement into the language of Fourier analysis. By the Fourier characterization of intersection bodies, we have that $(\|\cdot\|_K^{-1})^\wedge \geq 0$ on S^{n-1}. By (3), the condition that central hyperplane sections of L have greater volume than the corresponding sections of K is equivalent to $(\|\cdot\|_L^{-n+1})^\wedge \geq (\|\cdot\|_K^{-n+1})^\wedge$ everywhere on S^{n-1}. Therefore,

$$\int_{S^{n-1}} (\|\cdot\|_K^{-1})^\wedge(\xi)(\|\cdot\|_K^{-n+1})^\wedge(\xi)\, d\xi \leq \int_{S^{n-1}} (\|\cdot\|_K^{-1})^\wedge(\xi)(\|\cdot\|_L^{-n+1})^\wedge(\xi)\, d\xi.$$

By a version of Parseval's formula on the sphere, we can remove the Fourier transform in the latter inequality:

$$\int_{S^{n-1}} \|x\|_K^{-1}\|x\|_K^{-n+1}\, dx \leq \int_{S^{n-1}} \|x\|_K^{-1}\|x\|_L^{-n+1}\, dx.$$

Now the quantity in the left-hand side is equal to $n\mathrm{Vol}_n(K)$, and the expression in the right-hand side can be estimated from above using Hölder's inequality. We get

$$n\mathrm{Vol}_n(K) \leq \Big(\int_{S^{n-1}} \|x\|_K^{-n}\, dx\Big)^{1/n}\Big(\int_{S^{n-1}} \|x\|_L^{-n}\, dx\Big)^{(n-1)/n} =$$

$$\big(n\mathrm{Vol}_n(K)\big)^{1/n}\big(n\mathrm{Vol}_n(L)\big)^{(n-1)/n},$$

which implies $\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)$.

The affirmative part of Schneider's connection reads as follows: if L is a projection body, then for any symmetric convex body K the answer to the question of Shephard's problem is affirmative. Using (4) and the Fourier characterization of projection bodies, we formulate the latter statement as follows: if K and L are origin symmetric convex bodies in $\mathbb{R}^n$ with curvature functions $f_K,\ f_L$ and support functions

h_K, h_L satisfying $\widehat{h_L}(\theta) \le 0$ and $\widehat{f_K}(\theta) \ge \widehat{f_L}(\theta)$, $\forall \theta \in S^{n-1}$, then $\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)$.

To prove this statement, note that, by a version of Parseval's formula on the sphere,

$$\int_{S^{n-1}} \widehat{h_L}(\theta)\widehat{f_K}(\theta)d\theta \le \int_{S^{n-1}} \widehat{h_L}(\theta)\widehat{f_L}(\theta)d\theta$$

is equivalent to

$$\int_{S^{n-1}} h_L(\theta)f_K(\theta)d\theta \le \int_{S^{n-1}} h_L(\theta)f_L(\theta)d\theta.$$

Using integral representations of the mixed volume and volume (see [Ga3], page 354):

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(\theta)f_K(\theta)d\theta, \qquad \mathrm{Vol}_n(L) = \frac{1}{n}\int_{S^{n-1}} h_L(\theta)f_L(\theta)d\theta,$$

we get

$$V_1(K,L) \le \mathrm{Vol}_n(L)$$

Now, the well-known Minkowski's inequality,

$$V_1(K,L) \ge \mathrm{Vol}_n(L)^{\frac{1}{n}}\mathrm{Vol}_n(K)^{\frac{n-1}{n}},$$

immediately implies the result.

4. Appendix

Here we discuss the analog of formula (4) in the case of general convex bodies. We start with the definition of extended measures.

Let μ be a finite even Borel measure on S^{n-1}. A distribution μ_e is called the extended measure of μ if, for every even test function $\phi \in S(\mathbb{R}^n)$,

$$\langle \mu_e, \phi\rangle = \frac{1}{2}\int_{S^{n-1}} \langle r^{-2}, \phi(r\xi)\rangle d\mu(\xi). \tag{11}$$

In most cases we are only interested in test functions supported outside of the origin, for which $\langle r^{-2}, \phi(r\xi)\rangle = \int_{\mathbb{R}} r^{-2}\,\phi(r\xi)\,dr$. For the general definition see [GS].

It is shown in [KRZ], that for every $\theta \in S^{n-1}$

$$\widehat{\mu_e}(\theta) = -\frac{\pi}{2}\int_{S^{n-1}} |\theta\cdot y|d\mu(y). \tag{12}$$

An immediate consequence of the above identity is the following Fourier analytic formula for projections.

Let K be a convex origin symmetric body in $\mathbb{R}^n$ and let $S_e(K)$ be the extended measure of the surface area measure $S_{n-1}(K,\cdot)$. Then

$$\mathrm{Vol}_{n-1}\left(K\Big|\theta^\perp\right) = -\frac{1}{\pi}\widehat{S_e(K)}(\theta), \qquad \forall\theta\in S^{n-1}. \tag{13}$$

Another consequence is the Fourier analytic characterization of zonoids (projection bodies). We remind that a convex body L in $\mathbb{R}^n$ is called a zonoid, if its support function h_L is given by

$$h_L(x) = \frac{1}{2}\int_{S^{n-1}} |x\cdot\theta| d\mu(\theta).$$

Here μ is some positive, even Borel measure on S^{n-1}. The following fact may be found in [KRZ].

An origin symmetric convex body L in $\mathbb{R}^n$ is a zonoid if and only if there exists a measure μ on S^{n-1} so that

$$\widehat{h_L} = -(2\pi)^{n-1}\mu_e. \tag{14}$$

We conclude with a simple proof of the well-known Petty's volume formula for zonoids:

$$\mathrm{Vol}_n(L) = \frac{1}{n}\int_{S^{n-1}} \mathrm{Vol}_{n-1}(L|\theta^\perp) d\mu(\theta).$$

Indeed, Parseval's identity together with (12), (14), and the standard formula for the volume of convex body, give

$$\mathrm{Vol}_n(L) = \frac{1}{n}\int\limits_{S^{n-1}} h_L(\theta) dS_{n-1}(L,\theta) = \frac{1}{(2\pi)^n n}\int\limits_{S^{n-1}} \widehat{S_{n-1}(L,\cdot)}(\theta) d\widehat{h_L}(\theta) =$$

$$\frac{1}{n}\int\limits_{S^{n-1}} \mathrm{Vol}_{n-1}(L|\theta^\perp) d\mu(\theta).$$

Similarly, one can use (3) and Parseval's formula to get an expression for the volume of a body in terms of volumes of its central hyperplane sections (see [K7], Section 4):

$$vol_n(K) = \frac{(2\pi)^n\pi(n-1)}{n}\int_{S^{n-1}} vol_{n-1}(K\cap\xi^\perp)(\|\cdot\|_K^{-1})^\wedge(\xi)\ d\xi.$$

In the case, where K is an intersection body, $(\|\cdot\|_K^{-1})^\wedge$ represents a measure on S^{n-1}, so the latter formula is an analog of the formula for projections.

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