An Isomorphic Version of the Busemann–Petty Problem for Gaussian Measure

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Summary. In this paper we provide upper and lower bounds for the Gaussian Measure of a hyperplane section of a convex symmetric body. We use those estimates to give a partial answer to an isomorphic version of the Gaussian Busemann–Petty problem.

1 Introduction

The standard Gaussian measure on $\mathbb{R}^n$ is given by its density:

$$\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n} \int_K e^{-\frac{|x|^2}{2}} dx,$$

where $|x|^2 = \sum_{i=1}^n |x_i|^2$.
Consider two convex symmetric bodies (convex symmetric sets with nonempty interior) $K, L \subset \mathbb{R}^n$ such that

$$\gamma_{n-1}(K \cap \xi^\perp) \le \gamma_{n-1}(L \cap \xi^\perp), \ \ \forall \xi \in S^{n-1}, \tag{1}$$

where $K \cap \xi^\perp$ denotes the section of K by the hyperplane orthogonal to ξ. Does it follow that

$$\gamma_n(K) \le \gamma_n(L)?$$

This is a Gaussian analog of the Busemann–Petty problem (see [K] for details about the BP problem). It was shown in [Z], that the answer to the above question is affirmative if $n \le 4$ and it is negative if $n \ge 5$.

When $n \ge 5$, it is natural to ask an isomorphic version of the Gaussian Busemann–Petty problem: does there exist an absolute constant c so that if K and L are convex symmetric bodies satisfying (1) then $\gamma_n(K) \le c\gamma_n(L)$?

In this paper we give a partial answer to this question:

Theorem. *For any $n \in \mathbb{N}$ and convex symmetric bodies $K, L \subset \mathbb{R}^n$, if*

$$\gamma_{n-1}(K \cap \xi^\perp) \le \gamma_{n-1}(L \cap \xi^\perp), \ \ \forall \xi \in S^{n-1},$$

then

$$\gamma_n(K) \le c\,(\gamma_n(L))^{\frac{n-1}{n}}\,, \tag{2}$$

where c is an absolute constant.

This theorem will follow from two results proved below: the first one is that for any convex symmetric body K

$$\gamma_n(K) \le \gamma_{n-1}(K \cap \xi^\perp), \quad \forall \xi \in S^{n-1},$$

and the second one is that for any convex symmetric body L there exists a direction $\xi \in S^{n-1}$, such that

$$\gamma_{n-1}(L \cap \xi^\perp) \le c\big(\gamma_n(L)\big)^{\frac{n-1}{n}}. \tag{3}$$

It would be interesting to try to remove the power $(n-1)/n$ on the right hand side of (2), since it will then solve the same problem for the Lebesgue measure (the so-called slicing problem, see [MP1] for more details). We must note that (3) is sharp (consider a sequence of Euclidean balls with radii going to 0), so the theorem cannot be improved using the method described below.

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2 Proof of the Theorem

It was proved in [BS], that if $K \subset \mathbb{R}^n$ is a convex closed set with $\gamma_n(K) \ge \frac{1}{2}$, then $\gamma_{n-1}(K \cap \xi^\perp) \ge \frac{1}{2}$ for all $\xi \in S^{n-1}$. The next simple lemma shows that this result can be improved in the case of K being symmetric.

Lemma 1. *For any convex symmetric body $K \in \mathbb{R}^n$:*

$$\gamma_n(K) \le \gamma_{n-1}(K \cap \xi^\perp), \quad \forall \xi \in S^{n-1}.$$

Proof. By rotation invariance of the Gaussian measure, it is enough to consider the case $\xi = (0, \dots, 0, 1)$. Put

$$K(t) = K \cap \big((0, \cdots, 0, 1)^\perp + t(0, \cdots, 0, 1)\big)$$

and

$$PK(t) = K(t)\big|\mathbb{R}^{n-1} = \text{ orthogonal projection of } K(t) \text{ to } \mathbb{R}^{n-1}.$$

Note that K is a convex body, so for $0 < \lambda < 1$

$$\lambda K(a) + (1-\lambda)K(b) \subset K\big(\lambda a + (1-\lambda)b\big),$$

thus

$$\lambda PK(a) + (1-\lambda)PK(b) \subset PK\big(\lambda a + (1-\lambda)b\big).$$

In particular,

$$\frac{1}{2}PK(t) + \frac{1}{2}PK(-t) \subset PK(0) = K(0). \tag{4}$$

Next use that γ_{n-1} is a log-concave measure ([Bo], [E] or [Li]):

$$\log \gamma_{n-1}\big(\lambda A + (1-\lambda)B\big) \geq \lambda \log \gamma_{n-1}(A) + (1-\lambda)\log \gamma_{n-1}(B).$$

The latter inequality together with (4) gives

$$\begin{aligned}
\log \gamma_{n-1}\big(K(0)\big) &\geq \log \gamma_{n-1}\Big(\frac{1}{2}PK(t) + \frac{1}{2}PK(-t)\Big) \\
&\geq \frac{1}{2}\log \gamma_{n-1}\big(PK(t)\big) + \frac{1}{2}\log \gamma_{n-1}\big(PK(-t)\big) \\
&= \log \gamma_{n-1}\big(PK(t)\big).
\end{aligned}$$

So $\gamma_{n-1}(K(0)) \geq \gamma_{n-1}(PK(t))$.

Now we are ready to finish the proof:

$$\begin{aligned}
\gamma_n(K) &= \frac{1}{(\sqrt{2\pi})^n}\int_K e^{-\frac{|x|^2}{2}}dx \\
&= \frac{1}{(\sqrt{2\pi})^n}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\int_{K(t)} e^{-\frac{\sum_{i=1}^{n-1} x_i^2}{2}}dxdt \\
&= \frac{1}{(\sqrt{2\pi})^n}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\int_{PK(t)} e^{-\frac{\sum_{i=1}^{n-1} x_i^2}{2}}dxdt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\gamma_{n-1}\big(PK(t)\big)dt \\
&\leq \gamma_{n-1}\big(K(0)\big)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt = \gamma_{n-1}\big(K(0)\big).
\end{aligned} \tag{5}$$

□

Remark 1. It is clear that (5) actually shows:

$$\gamma_n(K) \leq \gamma_{n-1}(K \cap \xi^{\perp})\gamma_1(K\big|\xi), \quad \forall \xi \in S^{n-1}, \tag{6}$$

where $K\big|\xi$ is the orthogonal projection of K onto ξ.

Moreover it was pointed out to us by the referee, that (6) can be generalized to

$$\gamma_n(K) \leq \gamma_{n-k}(K \cap E^{\perp})\gamma_k(K\big|E), \tag{7}$$

for any k-dimensional subspace E of $\mathbb{R}^n$. Actually (7) is true for a general convex body K whose Gaussian-barycenter is 0: $\int_K x d\gamma_n(x) = 0$. This can be shown by straightforward generalization of a result of Milman and Pajor (see [MP2], Remark 2 on page 319 and Lemma 1 on page 316).

Our next goal is to find an upper bound for the Gaussian measure of a hyperplane section. First we will consider the case where K is a regular Euclidean ball of radius $t > 0$.

Lemma 2. *For any $t > 0$ and $n \in \mathbb{N}$,*

$$c_1\big(\gamma_n(tB_2^n)\big)^{\frac{n-1}{n}} \le \gamma_{n-1}(tB_2^{n-1}) \le c_2\big(\gamma_n(tB_2^n)\big)^{\frac{n-1}{n}}, \tag{8}$$

where $tB_2^n = \{x \in \mathbb{R}^n : |x| \le t\}$ and c_1, c_2 are absolute constants.

Proof. First note that for any $t \ge 0$

$$\int_0^t r^{n-2}e^{-\frac{r^2}{2}}dr \le \frac{n^{\frac{n-1}{n}}}{n-1}\left(\int_0^t r^{n-1}e^{-\frac{r^2}{2}}dr\right)^{\frac{n-1}{n}}. \tag{9}$$

To prove this inequality consider a function

$$f(t) = \int_0^t r^{n-2}e^{-\frac{r^2}{2}}dr - \frac{n^{\frac{n-1}{n}}}{n-1}\left(\int_0^t r^{n-1}e^{-\frac{r^2}{2}}dr\right)^{\frac{n-1}{n}},$$

then

$$\begin{aligned} f'(t) &= t^{n-2}e^{-\frac{t^2}{2}} - n^{-\frac{1}{n}}t^{n-1}e^{-\frac{t^2}{2}}\left(\int_0^t r^{n-1}e^{-\frac{r^2}{2}}dr\right)^{-\frac{1}{n}} \\ &= t^{n-2}e^{-\frac{t^2}{2}}\left(1 - n^{-\frac{1}{n}}t\Big(\int_0^t r^{n-1}e^{-\frac{r^2}{2}}dr\Big)^{-\frac{1}{n}}\right) \\ &< t^{n-2}e^{-\frac{t^2}{2}}\left(1 - n^{-\frac{1}{n}}t\Big(\int_0^t r^{n-1}dr\Big)^{-\frac{1}{n}}\right) = 0. \end{aligned}$$

Thus $f(t)$ is a decreasing function and $f(t) \le 0$ for $t > 0$, which proves the inequality (9).

Passing to polar coordinates and applying (9) we get

$$\begin{aligned} \gamma_{n-1}(tB_2^{n-1}) &= \frac{1}{(\sqrt{2\pi})^{n-1}}\int_{S^{n-2}}\int_0^t r^{n-2}e^{-\frac{r^2}{2}}dr\,d\theta \\ &= \frac{|S^{n-2}|}{(\sqrt{2\pi})^{n-1}}\int_0^t r^{n-2}e^{-\frac{r^2}{2}}dr \\ &\le \frac{|S^{n-2}|n^{\frac{n-1}{n}}}{(\sqrt{2\pi})^{n-1}(n-1)}\left(\int_0^t r^{n-1}e^{-\frac{r^2}{2}}dr\right)^{\frac{n-1}{n}} \\ &= \frac{|S^{n-2}|n^{\frac{n-1}{n}}}{|S^{n-1}|^{\frac{n-1}{n}}(n-1)}\big(\gamma_n(tB_2^n)\big)^{\frac{n-1}{n}}. \end{aligned} \tag{10}$$

Finally, from $|S^{n-1}| = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ and the Stirling asymptotic formula for the Gamma-function we get

$$\gamma_{n-1}(tB_2^{n-1}) \le c_2\big(\gamma_n(tB_2^n)\big)^{\frac{n-1}{n}}.$$

To prove the lower bound for $\gamma_{n-1}(tB_2^{n-1})$, we first note that if $t \ge 2\sqrt{n}$ then $\gamma_{n-1}(tB_2^{n-1}) > c$ (see [Li] or [Ba], p. 236), and thus

$$\gamma_{n-1}(tB_2^{n-1}) > c\big(\gamma_n(tB_2^n)\big)^{\frac{n-1}{n}}.$$

If $t < 2\sqrt{n}$ it is easy to show that

$$\left(\int_0^t r^{n-1}e^{-\frac{r^2}{2}}dr\right)^{\frac{n-1}{n}} < C\int_0^t r^{n-2}e^{-\frac{r^2}{2}}dr. \tag{11}$$

Indeed

$$\left(\int_0^t r^{n-1}e^{-\frac{r^2}{2}}dr\right)^{-\frac{1}{n}} \le \left(\int_0^t r^{n-1}e^{-2n}dr\right)^{-\frac{1}{n}} \le Ct^{-1}$$

and

$$\left(\int_0^t r^{n-1}e^{-\frac{r^2}{2}}dr\right)^{\frac{n-1}{n}} < C\int_0^t r^{n-1}t^{-1}e^{-\frac{r^2}{2}}dr < C\int_0^t r^{n-2}e^{-\frac{r^2}{2}}dr.$$

Finally, the lemma follows from (11) and computations similar to (10). □

To prove the next lemma we first recall the definition of an intersection body. Let K and M be origin symmetric star bodies in $\mathbb{R}^n$. Following Lutwak [Lu], we say that K is the *intersection body of* M if the radius of K in every direction is equal to the volume of the central hyperplane section of M perpendicular to this direction:

$$\|\xi\|_K^{-1} = \mathrm{Vol}_{n-1}(M \cap \xi^\perp), \ \ \forall \xi \in S^{n-1}.$$

A more general class of *intersection bodies* is defined as the closure in the radial metric of the class of intersection bodies of star bodies. It is clear that tB_2^n is an intersection body (see [Lu], [K] for more results and examples of intersection bodies). It was proved in [Z] (Theorems 3, 4), that if K is an intersection body then the answer to the Gaussian Busemann–Petty problem is affirmative for K and any convex symmetric body L, whose central sections have greater Gaussian measure than the corresponding sections of K.

Lemma 3. *For any convex symmetric body $K \subset \mathbb{R}^n$, there exists a direction $\xi \in S^{n-1}$ such that*

$$\gamma_{n-1}(K \cap \xi^\perp) \le c\big(\gamma_n(K)\big)^{\frac{n-1}{n}},$$

where c is an absolute constant.

Proof. Assume that for any $\xi \in S^{n-1}$

$$\gamma_{n-1}(K \cap \xi^{\perp}) > c_2\big(\gamma_n(K)\big)^{\frac{n-1}{n}},$$

where c_2 is the absolute constant from Lemma 1. Consider $t > 0$ so that

$$\gamma_{n-1}(tB_2^{n-1}) = c_2\big(\gamma_n(K)\big)^{\frac{n-1}{n}}. \tag{12}$$

Then

$$\gamma_{n-1}(K \cap \xi^{\perp}) > \gamma_{n-1}(tB_2^n \cap \xi^{\perp}), \ \ \forall \xi \in S^{n-1}. \tag{13}$$

But tB_2^n is an intersection body, therefore (13) implies

$$\gamma_n(K) > \gamma_n(tB_2^n).$$

The latter inequality together with (12) gives:

$$\big(c_2^{-1}\gamma_{n-1}(tB_2^{n-1})\big)^{\frac{n}{n-1}} > \gamma_n(tB^n)$$

or

$$\gamma_{n-1}(tB_2^{n-1}) > c_2\big(\gamma_n(tB_2^n)\big)^{\frac{n-1}{n}},$$

which contradicts Lemma 2. □

Remark 2. Using a similar method, one can show that for any intersection body $K \subset \mathbb{R}^n$, there exists a direction $\xi \in S^{n-1}$ such that

$$c_1\big(\gamma_n(K)\big)^{\frac{n-1}{n}} < \gamma_{n-1}(K \cap \xi^{\perp}),$$

where c_1 is the absolute constant from Lemma 1. Indeed, assuming the latter inequality not to be true, we get

$$\gamma_{n-1}(K \cap \xi^{\perp}) < \gamma_{n-1}(tB_2^n \cap \xi^{\perp}), \ \ \forall \xi \in S^{n-1},$$

where t is such that $\gamma_{n-1}(tB_2^{n-1}) = c_1\left(\gamma_n(K)\right)^{\frac{n-1}{n}}$. Again the Gaussian Busemann–Petty problem has an affirmative answer in this case, thus

$$c_1\big(\gamma_n(tB_2^n)\big)^{\frac{n-1}{n}} > \gamma_{n-1}(tB_2^{n-1}),$$

which again contradicts Lemma 2.

Proof of the Theorem. From Lemma 3 we get that there is $\xi \in S^{n-1}$ such that

$$\gamma_{n-1}(L \cap \xi^{\perp}) \le c\big(\gamma_n(L)\big)^{\frac{n-1}{n}},$$

but then, from Lemma 1:

$$\gamma_n(K) \le \gamma_{n-1}(K \cap \xi^{\perp}).$$

Using

$$\gamma_{n-1}(K \cap \xi^{\perp}) \le \gamma_{n-1}(L \cap \xi^{\perp}), \quad \forall \xi \in S^{n-1},$$

we get

$$\gamma_n(K) \le c\big(\gamma_n(L)\big)^{\frac{n-1}{n}}.$$

□

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