GAUSSIAN MEASURE OF SECTIONS OF DILATES AND TRANSLATIONS OF CONVEX BODIES.

A. ZVAVITCH

Abstract. In this paper we give a solution for the Gaussian version of the Busemann-Petty problem with additional information about dilates and translations. We also make a remark on the size of the Gaussian measure of the dilates of the unit cube.

1. Introduction

The standard Gaussian measure on $\mathbb{R}^n$ is given by its density:

$$\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n} \int_K e^{-\frac{|x|^2}{2}} dx,$$

where $|x|^2 = \sum_{i=1}^n |x_i|^2$. We refer to [Bog] for general properties and facts.

Consider two convex symmetric bodies (convex, compact, symmetric sets with nonempty interior) $K, L \subset \mathbb{R}^n$ such that

$$\gamma_{n-1}(K \cap \xi^\perp) \leq \gamma_{n-1}(L \cap \xi^\perp), \ \ \forall \xi \in S^{n-1}, \tag{1}$$

where $K \cap \xi^\perp$ denotes the section of K by the hyperplane orthogonal to ξ. Does it follow that

$$\gamma_n(K) \leq \gamma_n(L)?$$

This is a Gaussian analog of the Busemann-Petty problem (see [K], [Ga] or [Lu], [GKS], [Zh] for details about the Busemann-Petty problem for volume measure). It was shown in [Z1], that the answer to the above question is affirmative if $n \leq 4$ and it is negative if $n \geq 5$ (is was also shown in [Z2] that the same theorem is true for a general class of measures). This leads to the following questions posed to the author by V. Milman: will the answer to the problem change into the positive direction if we compare not only the Gaussian measure of sections of the bodies but also the Gaussian measure of section of their dilates?

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What will happen if we would compare the translations of Gaussian measure of each section? More precisely:

The Dilation Problem: *Consider two convex symmetric bodies $K, L \subset \mathbb{R}^n$ such that*

$$\gamma_{n-1}(rK \cap \xi^\perp) \leq \gamma_{n-1}(rL \cap \xi^\perp), \ \ \forall \xi \in S^{n-1} \text{ and } \forall r > 0, \qquad (2)$$

Does it follow that

$$\gamma_n(K) \leq \gamma_n(L)?$$

The idea to consider the dilations of convex bodies seems to be a natural one. The Gaussian measure is not a homogenous measure and there is no trivial connection between $\gamma_n(rK)$ and $\gamma_n(K)$. A number of extremely interesting properties of Gaussian measure of dilates of symmetric convex sets were recently proved (see [LO1], [CFM], [LO2]). In addition, it has been commonly observed that the introduction of an additional parameter or integral in Busemann-Petty type problems leads to a positive answer in the higher dimension (see [K1], [K2], [RZ], [Z2], [K]).

In Section 1 we show that although inequalities on dilations add some strength to the condition on the bodies, nevertheless the answer to the Dilation Problem is negative for $n \geq 7$. Note that this leaves the Dilation Problem open in dimensions $5, 6$ and the technique presented below does require a crucial modification to provide a solution for $n = 5, 6$.

The Translation Problem: *Consider two convex symmetric bodies $K, L \subset \mathbb{R}^n$ such that*

$$\gamma_{n-1}(\left[K \cap \xi^\perp\right] + v) \leq \gamma_{n-1}(\left[L \cap \xi^\perp\right] + v), \ \ \forall \xi \in S^{n-1} \text{ and } \forall v \in \xi^\perp,$$

Does it follow that

$$\gamma_n(K) \leq \gamma_n(L)?$$

In Section 3 we present an answer to the translation problem. We show that the answer is affirmative in all dimensions, in fact, we observe that a much stronger property holds.

We say that a closed set K in $\mathbb{R}^n$ is a **star body** if every straight line passing through the origin crosses the boundary of K at exactly two points, the origin is an interior point of K and the boundary of K is continuous.

In Theorem 2 we prove that if $K, L \subset \mathbb{R}^n$ are star bodies, such that L is convex and

$$\gamma_n(K + v) \leq \gamma_n(L + v), \ \ \forall v \in \mathbb{R}^n,$$

then $K \subseteq L$. We also show that the convexity assumption of L is necessary.

In addition to the above questions in Section 2 we apply the large deviation principle for Gaussian measure to show the asymptotic sharpness of the recent results from [BGMN] on the Gaussian measure of the hyperplane sections of the dilates of the unit cube.

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2. The Dilation Problem

We say that K is the intersection body of M if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the central hyperplane section of M perpendicular to this direction. A more general class of *intersection bodies* is defined as the closure in the radial metric of the class of intersection bodies of star bodies (see [Lu], [Ga] and [K] for precise definition and properties).

It was proved in [Z1] that if K from (1) is an intersection body then the answer to the Gaussian Busemann-Petty problem is affirmative for any star-shaped body L. Thus we may immediately conclude that the dilation problem also has an affirmative answer in this case; in particular, the answer is affirmative when K is a dilate of a centered Euclidean ball (this may be also shown by the simple averaging argument, together with an elementary inequality proved in [Z1]).

Lemma 1. *Consider two star bodies $K, L \subseteq \mathbb{R}^n$ and suppose that K is a dilate of an Euclidean ball. Then from*

$$\gamma_{n-1}(rK \cap \xi^{\perp}) \leq \gamma_{n-1}(rL \cap \xi^{\perp}), \ \ \forall \xi \in S^{n-1}, \ \ \forall r > 0, \tag{3}$$

it follows that $K \subseteq L$.

Proof : Let us start with some definitions and notations. For a Borel set $A \subseteq \mathbb{R}^n$ denote by A° the interior of A and by $\bar{A}$ its closure. As usual B_2^n denotes the Euclidean ball in $\mathbb{R}^n$, i.e. $B_2^n = \{x \in \mathbb{R}^n : |x| = 1\}$ and

$$I(A) = \frac{1}{2} \inf_{x \in A} |x|^2.$$

The following chain of inequalities represents the classical large deviation principle (Corollary 4.9.3 in [Bog]):

$$\begin{aligned} -I(A^{\circ}) &\leq \liminf_{\varepsilon \to 0} \varepsilon^2 \log \gamma_n(\frac{1}{\varepsilon} A) \\ &\leq \limsup_{\varepsilon \to 0} \varepsilon^2 \log \gamma_n(\frac{1}{\varepsilon} A) \leq -I(\bar{A}). \end{aligned} \tag{4}$$

For a star body K we denote $K^c = \mathbb{R}^n \setminus K$, thus $K^{c,\circ}$ denotes the interior of the compliment of K. Using compactness of K we get

$$I(K^{c,\circ}) = I(\bar{K^c}) = \frac{1}{2}w(K)^2, \tag{5}$$

where $w(K) = \sup\{r > 0 : rB_2^n \subseteq K\}$ is the inradius of K (we refer to [LO2] for more connections of properties of inradius and Gaussian measure). Thus applying (4) and (5) with $A = K^c$, we conclude

$$\lim_{r\to\infty} \frac{1}{r^2} \log \gamma_n(rK^c) = -\frac{1}{2}w(K)^2. \tag{6}$$

Now we are ready to finish the proof of the lemma, indeed, from (3) we obtain

$$\gamma_{n-1}(rK^c \cap \xi^\perp) \geq \gamma_{n-1}(rL^c \cap \xi^\perp), \ \ \forall r > 0, \ \ \forall \xi \in S^{n-1},$$

and, applying (6), we conclude

$$w(K \cap \xi^\perp) \leq w(L \cap \xi^\perp), \ \ \forall \xi \in S^{n-1}.$$

Note that in the case $K = tB_2^n$ the above inequality implies $(K \cap \xi^\perp) \subseteq (L \cap \xi^\perp)$ for all $\xi \in S^{n-1}$ and thus $K \subseteq L$.

□

Next we will show that in general case the the Dilation Problem has a negative answer for $n \geq 7$. The main idea of the proof is to relate the Dilation Problem with the original Busemann-Petty problem for volume measure. We will also use one dimensional version of Anderson's inequality for Gaussian measure (see [A] or [Bog], p. 28):

$$\gamma_1([-a, a]) \geq \gamma_1([-a + b, a + b]), \forall a > 0 \text{ and } b \in \mathbb{R}. \tag{7}$$

Theorem 1. *There are convex symmetric bodies $K, L \subset \mathbb{R}^n$, $n \geq 7$, such that*

$$\gamma_{n-1}(rK \cap \xi^\perp) \leq \gamma_{n-1}(rL \cap \xi^\perp), \ \ \forall \xi \in S^{n-1} \ \textit{and} \ \forall r > 0,$$

but $\gamma_n(K) > \gamma_n(L)$.

Proof : We denote by Vol_n the usual n-dimensional volume (Lebesgue) measure on $\mathbb{R}^n$.

Assume that the Dilation Problem has an affirmative answer in $\mathbb{R}^n$ for some fixed n, then for any K and L such that

$$\gamma_{n-1}(rK \cap \xi^\perp) \leq \gamma_{n-1}(rL \cap \xi^\perp), \ \ \forall \xi \in S^{n-1} \text{ and } \forall r > 0, \tag{8}$$

we would obtain $\gamma_n(K) \leq \gamma_n(L)$. Note that the condition on sections (8) will be also satisfied for bodies tK and tL, for all $t > 0$. Thus we conclude $\gamma_n(tK) \leq \gamma_n(tL)$, for all $t > 0$.

Writing the Gaussian measure as an integral and making the change of variables we get

$$\int_K e^{-\frac{t^2|x|^2}{2}}dx \leq \int_L e^{-\frac{t^2|x|^2}{2}}dx.$$

Taking the limits as t goes to 0 from the both sides of the above inequality, we obtain

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

Thus the affirmative answer to the Dilation Problem would also imply that if

$$\gamma_{n-1}(rK \cap \xi^\perp) \leq \gamma_{n-1}(rL \cap \xi^\perp), \ \ \forall \xi \in S^{n-1} \text{ and } \forall r > 0,$$

then $\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)$.

In [Bo], Bourgain constructed a counterexample to the original Busemann-Petty problem for volume measure in $\mathbb{R}^n$ for $n \geq 7$. Bourgain provided an example of $K \subset \mathbb{R}^n$, such that

$$\mathrm{Vol}_{n-1}(K \cap \xi^\perp) \leq \mathrm{Vol}_{n-1}(B_2^n \cap \xi^\perp), \quad \forall \xi \in S^{n-1} \tag{9}$$

but $\mathrm{Vol}_n(K) > \mathrm{Vol}_n(B_2^n)$. We will show now that the same counterexample will work the Dilation Problem. Due to the homogeneity property of the volume measure the condition on sections is also true for dilates of K and B_2^n. The last step is the following well known fact **Fact:** *If K is a convex body in $\mathbb{R}^n$ and $Vol_n(K) = Vol_n(tB_2^n)$ then $\gamma_n(K) \leq \gamma_n(tB_2^n)$.*

This claim can be proved by applying the Steiner symmetrizations to a body K (see [W], p. 306). Indeed, the volume does not change under Steiner symmetrizations, but the Gaussian measure will increase due to the Anderson's inequality (7).

Applying the above fact to the dilates of $K \cap \xi^\perp$ we get from (9):

$$\gamma_{n-1}(rK \cap \xi^\perp) \leq \gamma_{n-1}(rB_2^n \cap \xi^\perp), \ \ \forall \xi \in S^{n-1} \text{ and } \forall r > 0,$$

but $\mathrm{Vol}_n(K) > \mathrm{Vol}_n(B_2^n)$.

□

3. Gaussian measure of hyperplane sections of dilates of the unit cube

Let $B_\infty^n = \{x \in \mathbb{R}^n : \max_{1\leq i\leq n} |x_i| \leq 1\}$. The famous Ball's slicing theorem [Ba] stays that

$$\mathrm{Vol}_{n-1}(B_\infty^n \cap \xi^\perp) \leq \mathrm{Vol}_{n-1}(B_\infty^n \cap \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\}^\perp), \ \ \forall \xi \in S^{n-1}.$$

It is an open question to find an analogous result for the Gaussian measure. The following estimate was proved in [BGMN]:

$$\gamma_{n-1}(rB_\infty^n \cap \xi^\perp) \leq \gamma_{n-1}(r\sqrt{\frac{n}{n-1}}B_\infty^{n-1}), \ \ \forall \xi \in S^{n-1}, \ \forall r > 0.$$

Here we will observe that the above result is "asymptotically" sharp (as stated above, i.e. for all $r > 0$) and the choice of a maximal hyperplane for the dilates of B_∞^n depends on r. Indeed, when r is small enough the Gaussian measure is close to the volume measure and the answer is the same as in volume case, i.e., $\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\}^\perp$.

Next consider $r \to \infty$. Notice that

$$w(B_\infty^n \cap \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\}^\perp) = 1. \tag{10}$$

Also notice that

$$w(B_\infty^n \cap \{\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\}^\perp) = \sqrt{\frac{n}{n-1}}, \tag{11}$$

indeed, $(-1, 1/(n-1), \cdots, 1/(n-1))$ will be a tangent point of the maximal inscribed ball to the boundary of $B_\infty^n \cap \{\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\}^\perp$.

Finally, using the large deviation principle (4) and equality (6) from the previous section, together with (10) and (11) from above, we get that $\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\}^\perp$ will not longer be a maximal hyperplane sections of rB_∞^n for r large enough and that the result in [BGMN] is sharp for $r \to \infty$.

4. Translations

Theorem 2. *Consider two star bodies $K, L \subset \mathbb{R}^n$, and assume that L is convex and*

$$\gamma_n(K + v) \leq \gamma_n(L + v), \ \ \forall v \in \mathbb{R}^n. \tag{12}$$

Then $K \subseteq L$.

Proof : Assume that the theorem is not true and that there are sets K and L satisfying assumptions of the theorem but such that $K \not\subseteq L$. Then we may apply convexity argument to observe that there is a hyperplane separating L from a set $K'' \subset K$. Note that K is a star body, thus we may assume that K'' is an open set. More precisely, there is an open subset K'' of K, a vector $u \in S^{n-1}$ and a positive number a so that

$$L \subset \{x : x \cdot u < a\} \text{ but } K'' \subset \{x : x \cdot u > a\}.$$

Note that inequality (12) is also true for sets K'' and $L' \equiv \{x : x \cdot u \leq a\}$. In addition, we consider an orthogonal box $K' \subset K''$, such that one of the sides of K' is parallel to the vector v, inequality (12) is true for K' and L'.

Gaussian measure is rotation invariant, thus we may assume that $u = (-1, 0, \ldots, 0)$ and

$$L' = \{x : x_1 \geq -a\} \text{ and } K' = [b, -a] \times \prod_{i=2}^{n} [b_i, a_i],$$

where $b < -a < 0$. Using (12) for $v = (t, 0, \ldots, 0)$, $t > 0$, we conclude

$$C \int_{b+t}^{-a+t} e^{-\frac{x^2}{2}} dx \leq \int_{-a+t}^{\infty} e^{-\frac{x^2}{2}} dx, \ \ \forall t > 0,$$

where $C = \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} e^{-\frac{|x|^2}{2}} dx_2 \cdots dx_n > 0$ is a constant independent of t. Adding $C \int_{-a+t}^{\infty} e^{-\frac{x^2}{2}} dx$ to the both sides of the above inequality we get

$$C \int_{b+t}^{\infty} e^{-\frac{x^2}{2}} dx \leq (C+1) \int_{-a+t}^{\infty} e^{-\frac{x^2}{2}} dx, \ \ \forall t > 0$$

or

$$\frac{C}{C+1} \int_{b+t}^{\infty} e^{-\frac{x^2}{2}} dx \leq \int_{-a+t}^{\infty} e^{-\frac{x^2}{2}} dx, \ \ \forall t > 0. \tag{13}$$

Now we will use the standard bound for the tails of Gaussian distribution (see [Bog] page 2):

$$\left(\frac{1}{y} - \frac{1}{y^3}\right) e^{-\frac{y^2}{2}} \leq \int_{y}^{\infty} e^{-\frac{x^2}{2}} dx \leq \frac{1}{y} e^{-\frac{y^2}{2}}, \ \ \forall y > 0. \tag{14}$$

Applying (14) to (13) we conclude, for t large enough:

$$\frac{C}{C+1} \left(\frac{1}{b+t} - \frac{1}{(b+t)^3}\right) e^{-\frac{(b+t)^2}{2}} \leq \frac{1}{-a+t} e^{-\frac{(-a+t)^2}{2}}.$$

Finally, from the last inequality with $b < -a$ we obtain a contradiction as $t \to \infty$.

□

Remark 1: Clearly, the similar result with almost identical prove will be true for a wide class of measures. For example, for any rotation invariant measure whose density decays fast enough at infinity. We also note that the result will be true for a more general classes of bodies K, for example, K and L may be assumed unbounded.

Remark 2: *The convexity assumption on L is necessary in the above theorem.* Indeed, one can construct a counterexample consisting of two star sets in $\mathbb{R}^2$. Take

$$K \equiv B_\infty^2 = \{(x,y) : |x|, |y| \leq 1\} \text{ and } L \equiv \{(x,y) : b|x| \leq |y|\},$$

where $b > 0$. Clearly $K \not\subseteq L$ and our goal is to show that we may choose b such that K and L would satisfy (12). First, we choose small b which would guarantee (12) for $v = (t, 0)$.

Note that $\lim_{b\to 0} \gamma_2(L + (0,t)) = 1$, for fixed t. Using this property we choose small b so that

$$\gamma_2(K + (t,0)) < \gamma_2(L + (t,0)), \text{ for all } |t| \leq 2.$$

For $|t| > 2$ the required inequality (12) follows immediately from rotation invariance of the Gaussian measure. Indeed, rotate the set $K + (t,0)$ by $\pi/2$ (this will not change its Gaussian measure) and noticing that $\{(x,y) : |x| \leq 1, -1 + t \leq y \leq 1 + t\} \subset L + (t,0)$ for $|t| \geq 2$ and $b < 1/3$.

To finish the construction we observe that $K + (t,0)$ and $L + (t,0)$ are symmetric with respect to x-axis. Thus, applying Anderson's inequality (7) to $K + (t,0)$ and $(L + (t,0))^c$, we obtain

$$\gamma_2(K + (t,0) + (0,d)) \leq \gamma_2(K + (t,0)) \quad \text{and}$$
$$\gamma_2(L + (t,0)) \leq \gamma_2(L + (t,0) + (0,d)).$$

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A. Zvavitch, The Department of Mathematical Sciences, Mathematics and Computer Science Building, Summit Street, Kent OH 44242, USA

E-mail address: zvavitch@math.kent.edu