1. Introduction

My research is devoted to exploring geometrical problems using tools from Analysis and Probability. In particular, I have worked on isoperimetric type questions in high dimensions. The classical Isoperimetric inequality is a well-known fact which dates back to the 19th century. It states that of all the sets of fixed volume the ball has the smallest surface area. Nowadays, dozens of generalizations and companions of this fact have been developed. An especially interesting one is the Gaussian Isoperimetric Inequality, obtained independently by Sudakov and Tsirelson [38] in 1974 and Borell [7] in 1975. It states that among all the sets in \( \mathbb{R}^n \) with prescribed Gaussian measure, half spaces have the smallest Gaussian surface area. By Gaussian surface area here we understand the Gaussian density averaged against the surface measure of the set.

Mushtari and Kwapien asked the reverse question: how large can the Gaussian perimeter of a convex set in \( \mathbb{R}^n \) be? In the literature this
question has been referred to as the “reverse isoperimetric problem”. Ball [2] proved in 1993 that the Gaussian perimeter of a convex set in \( \mathbb{R}^n \) is bounded from above by \( Cn^{\frac{1}{4}} \), where \( C \) is an absolute constant independent of the dimension. Nazarov [32] showed in 2003 that this bound is asymptotically exact. Further estimates for the special case of polynomial level set surfaces were provided by Kane [17].

In [25], [26] and [27] I have studied the question of Mushtari and Kwapien for more general classes of measures.

- In [25], I considered the probability measure \( \gamma_p \) with density \( C_{n,p}e^{-\frac{|x|^p}{p}} \), where \( p > 0 \). I have shown that the maximal perimeter of a convex set in \( \mathbb{R}^n \) with respect to \( \gamma_p \) is of order \( C(p)n^{\frac{3}{4} - \frac{1}{p}} \), where \( C(p) \) is a constant independent of the dimension.

- In [26], I solved the reverse isoperimetric problem for the entire class of log-concave rotation invariant probability measures (roughly speaking, the measure is log-concave if the logarithm of its density is a concave function). The latter class includes all the measures \( \gamma_p \) when \( p \geq 1 \), and, in particular, the standard Gaussian measure (the case \( p = 2 \)). It also includes the Lebesgue measure restricted to a ball.

- In [27], I carried out a more refined study of the maximal perimeter of polytopes in \( \mathbb{R}^n \) depending on the number of their facets.

There are several connections between the questions related to surface areas and problems from different areas of mathematics, such as Random Processes (see, for example, Ding, Eldan, Zhai [12]), Probability Theory (see, for example, Bentkus [3]), Signal Processing (see, for example, Kane [17]) and other areas.

My other work in Asymptotic Analysis includes an improvement of the Gaussian concentration of measure inequality. Roughly speaking, the classical Gaussian concentration inequality states that the neighborhood of a set has smaller Gaussian measure whenever the set is a half space (see Section 4 below for the precise formulation). This fact is used in many areas of mathematics (see an influential book of Ledoux, Talagrand [24], as well as the work of Rudelson and Vershynin [35], [36]). In [28], I have investigated the relation between the Gaussian concentration and the Gaussian surface area of a convex set.

Another direction of my research is devoted to the Gaussian Brunn-Minkowski inequality. This inequality appeared as a Conjecture in the work of Gardner and Zvavitch [16]. One of the forms of the classical Brunn-Minkowski inequality (see, for example, Schneider [37], Gardner [15] and Eldan, Klartag [14]) states that for any bounded Borel
measurable \( A, B \subset \mathbb{R}^n \) and for any \( \lambda \in [0, 1] \),

\[
|\lambda A + (1 - \lambda)B|^{\frac{1}{n}} \geq \lambda |A|^{\frac{1}{n}} + (1 - \lambda)|B|^{\frac{1}{n}},
\]

where \( | \cdot | \) stands for the Lebesgue Measure. Gardner and Zvavitch conjectured in [16] that for the standard Gaussian measure \( \gamma_2 \) the same inequality holds under some natural assumptions on the sets \( A \) and \( B \). Some positive results about the inequality

\[
\gamma_2(\lambda A + (1 - \lambda)B)^{\frac{1}{2}} \geq \lambda \gamma_2(A)^{\frac{1}{2}} + (1 - \lambda)\gamma_2(B)^{\frac{1}{2}}
\]

were obtained in [16] and later generalized by Marsiglietti [29]. However, (1) is false in the full generality: one may shift the set \( A \) away from the origin. The farther the shift, the smaller the right hand side of (1) becomes, while the left hand side stays bounded from below by the fixed quantity \( (1 - \lambda)\gamma_2(B)^{\frac{1}{2}} \).

One of the questions in [16] stated: Is (1) true for all the convex sets \( A \) and \( B \) containing the origin? The negative answer was obtained by Tkocz and Nayar [31].

It was also conjectured in both [16] and [31] that the inequality (1) holds true for all symmetric sets. Recently, I found a functional criterion for the inequality (1) in \( \mathbb{R}^2 \) for symmetric sets. I also showed that the inequality (1) is true on the plane in a neighborhood of any disc.

These advances were made possible by a formula (see Corollary 5.3 below) that I derived, which expresses the Gaussian measure of a convex set in \( \mathbb{R}^2 \) in terms of the support function of the set (the support function of a set \( Q \) is defined on the unit sphere as the distance to the support hyperplane of \( Q \) in the given direction). An analogous formula (see Proposition 5.2 below) also holds for any absolutely continuous measure on \( \mathbb{R}^n \).

In the following sections I describe the specifics of my research in more detail. Along the way, I also discuss my current work and some open problems.

2. Maximal surface area of a convex set in \( \mathbb{R}^n \) with respect to some classes of measures.

A measure \( \gamma \) on \( \mathbb{R}^n \) is called log-concave if for any Borel measurable sets \( A, B \subset \mathbb{R}^n \) and for any \( \lambda \in [0, 1] \),

\[
\gamma(\lambda A + (1 - \lambda)B) \geq \gamma(A)^{\lambda} \gamma(B)^{1-\lambda}.
\]

Here

\[ A + B = \{a + b \mid a \in A, b \in B\} \]
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is the Minkowski sum of the sets $A$ and $B$, and

$$\lambda A := \{ \lambda a \mid a \in A \}.$$ 

Log-concave measures have been studied intensively in the recent years. For the background and numerous interesting properties, see, for example, results of Klartag [20], [21], Milman [22], Pajor [30], Adamczak, Latala, Litvak, Tomczak-Jaegermann [11], Paouris [33].

A measure $\gamma$ is called rotation invariant if for every rotation $T$ and for every set $A$,

$$\gamma(TA) = \gamma(A).$$ 

Log-concave rotation invariant measures have been studied, in particular, by Bobkov in [4], [5], [6].

Examples of log-concave rotation invariant probability measures include the Standard Gaussian Measure $\gamma_2$ and Lebesgue measure restricted to a ball.

The surface area of a convex set $Q$ with respect to the measure $\gamma$ is defined to be

$$\gamma(\partial Q) = \liminf_{\epsilon \to +0} \frac{\gamma((Q + \epsilon B_2^n) \setminus Q)}{\epsilon},$$

where $B_2^n$ denotes Euclidian ball in $\mathbb{R}^n$.

I have studied questions related to the Surface Area of a convex set in $\mathbb{R}^n$ with respect to log-concave rotation invariant probability measures. In [26] I proved the following theorem.

**Theorem 2.1.** Fix $n \geq 2$. Let $\gamma$ be a log-concave rotation invariant probability measure on $\mathbb{R}^n$. Consider a random vector $X$ in $\mathbb{R}^n$ distributed with respect to $\gamma$. Then

$$\max_Q \gamma(\partial Q) \approx \frac{\sqrt{n}}{\sqrt{\mathbb{E}|X|} \sqrt{\text{Var}|X|}},$$

where $\mathbb{E}|X|$ and $\text{Var}|X|$ stand for the expectation and the variance of $|X|$ respectively. The maximum runs over the class of convex sets in $\mathbb{R}^n$. The notation “$\approx$” means that the equality is asymptotic.

In [25] I considered the measures $\gamma_p$ with densities $C_{n,p} e^{-\frac{|x|^p}{p}}$, where $p > 0$ and $C_{n,p}$ is the normalizing constant. Such measures are log-concave for $p \geq 1$, but not for $p \in (0, 1)$. In [25] I proved the following result.

**Theorem 2.2.** Let $n \geq 2$. Then

$$c(p)n^{\frac{3}{4} - \frac{1}{p}} \leq \max_Q \gamma_p(\partial Q) \leq C(p)n^{\frac{3}{4} - \frac{1}{p}},$$
where \(c(p)\) and \(C(p)\) are constants depending on \(p\) only, and the maximum runs over the class of convex sets in \(\mathbb{R}^n\).

Theorem 2.2 is a direct consequence of Theorem 2.1 in the case of \(p \geq 1\).

2.1. Current work and open problems. An interesting problem is to generalize Theorem 2.1 to the case of all log-concave probability measures. However, the asymptotic equality should in this case involve additional parameters. Indeed, if \(\gamma\) is the normalized Lebesgue measure on a cube (which is log-concave), the conclusion of Theorem 2.1 does not hold: in that case the maximal surface area is \(2n\), but \(E|X| \approx \sqrt{n}\) and \(Var|X| \approx 1\) (see the book by Brazitikos, Giannopoulos, Valettas, Vritsiou [8]).

Another problem I am working on is to get tight bounds on the maximal surface area of a convex set \(Q\) in \(\mathbb{R}^n\) with respect to a measure \(\gamma(Q)\). I was able to obtain certain bounds from above and below, but these bounds are not tight for every value of \(\gamma(Q)\). My approach to this problem relies on results obtained by Paouris and Pivovarov in [34].

I am also working on the exact reverse isoperimetric problem for the standard Gaussian measure on the plane: find the convex set in \(\mathbb{R}^2\) with the largest Gaussian surface area. That question arose from the discussion with Amir Livne Bar-on. The conjecture is that it is a “very thin” strip. I am trying to apply variational methods to that problem and use the representation of the Gaussian surface area of the set \(Q\) in terms of the support function of \(Q\).

I am working on similar problems related to different notions of the surface area and have obtained some estimates for them as well. In addition to that I have been considering different isoperimetric type questions, including versions of the Kannan-Lovasz-Simonovits-conjecture (see Kannan, Lovasz, Simonovits [19], Eldan [13]), and the isoperimetric inequality for rotation invariant log-concave probability measures.

3. Maximal surface area of a convex polytope in \(\mathbb{R}^n\) with respect to some classes of measures.

I have been working on the reverse isoperimetric questions for the class of convex polytopes in \(\mathbb{R}^n\) with \(K\) facets. In [27] I proved the following theorem.

**Theorem 3.1.** Fix \(n \geq 2\). Let \(\gamma\) be a log-concave rotation invariant probability measure on \(\mathbb{R}^n\). Consider a random vector \(X\) in \(\mathbb{R}^n\) distributed with respect to \(\gamma\). By \(E|X|\) and \(Var|X|\) denote the expectation...
and variance of $|X|$. Fix a positive integer $K \in [1, e^{\frac{\mathbb{E}|X|}{\sqrt{\text{Var}|X|}}}]$. Let $P$ be a convex polytope in $\mathbb{R}^n$ with $K$ facets. Then
\[
\gamma(\partial P) \leq C \frac{\sqrt{n}}{\mathbb{E}|X|} \cdot \sqrt{\log K} \cdot \log n,
\]
where $C$ and $c$ stand for absolute constants, independent of $P$, $\gamma$ and $n$.

Theorem 3.1 up to a factor of $\log n$, is a generalization of Nazarov’s estimate for the standard Gaussian measure. For a generalization of the same result by Nazarov in an entirely different set up see Kane [18].

I have shown as well, that Theorem 3.1 is sharp up to a factor of $\log n$, which was new even for the standard Gaussian measure.

**Theorem 3.2.** Fix $n \geq 2$. Let $\gamma$ be a log-concave rotation invariant probability measure on $\mathbb{R}^n$. Consider a random vector $X$ in $\mathbb{R}^n$ distributed with respect to $\gamma$. By $\mathbb{E}|X|$ and $\text{Var}|X|$ denote the expectation and variance of $|X|$. Fix a positive integer $K \in [1, e^{\frac{\mathbb{E}|X|}{\sqrt{\text{Var}|X|}}}]$. Then there exists a convex polytope $P$ in $\mathbb{R}^n$ with at most $K$ facets such that
\[
\gamma(\partial P) \geq C' \frac{\sqrt{n}}{\mathbb{E}|X|} \sqrt{\log K},
\]
where $c$ and $C'$ stand for absolute constants, independent of $P$, $\gamma$ and $n$.

For the measures $\gamma_p$ I have shown [27] the exact upper bound for the maximal surface area of a convex polytope in $\mathbb{R}^n$ with $K$ facets:

**Theorem 3.3.** Fix $n \geq 2$ and $p > 0$. Let $\gamma_p$ be the probability measure on $\mathbb{R}^n$ with density $C(n, p) e^{-\frac{|x|^p}{n}}$. Consider a random vector $X$ in $\mathbb{R}^n$ distributed with respect to $\gamma_p$. By $\mathbb{E}|X|$ denote the expectation of $|X|$. Fix a positive integer $K$. Let $P$ be a convex polytope in $\mathbb{R}^n$ with $K$ facets. Then
\[
\gamma_p(\partial P) \leq C(p) \frac{\sqrt{n}}{\mathbb{E}|X|} \sqrt{\log K} = C'(p)n^{\frac{1}{2} - \frac{1}{p}} \sqrt{\log K},
\]
where $C(p)$ and $C'(p)$ are constants depending on $p$ only.

Theorem 3.3 is asymptotically sharp since it matches the lower bound from Theorem 3.2.
3.1. Current work and open problems. In my future work I am hoping to find an approach to this problem which would eliminate the logarithmic discrepancy between Theorems 3.1 and 3.2.

Also, I have been working on a problem suggested by Nazarov: find analogous estimates for convex sets which are obtained as an intersection of polynomial level sets, in terms of the number of the level sets and the degree of the polynomial. The usual polytope is an intersection of half spaces, i.e. the level sets of polynomials of degree 1. It would be very interesting to obtain some reasonable estimates for the surface area of $Q$ which is an intersection of a given number of balls, i.e. the level sets of polynomials of degree 2. This problem is much more difficult since it is not clear how to estimate the surface area of the shifted polynomial boundary. Some ideas I have about it include approximation of norms induced by polynomial convex bodies with the norms of $l_\infty$-type.

4. On some improvements of the Gaussian concentration inequality in terms of the Gaussian surface area.

Let $(X, \mu)$ be a compact metric space with a Borel probability measure $\mu$. Let $B \subset X$ be a unit ball. The concentration function $\alpha(X, h)$ is defined to be

$$\alpha(X, h) = \sup_{\mu(A) \geq \frac{1}{2}} (1 - \mu(A + hB)),$$

where $A$ is always a Borel subset of $X$ (see Ledoux, Talagrand [24], page 16).

Analogously, for a measurable set $Q \subset \mathbb{R}^n$ we define a function

$$\alpha_Q(h) : \mathbb{R}^+ \to \mathbb{R}$$

by

$$\alpha_Q(h) := 1 - \gamma_2(Q + hB^n_2).$$

Here $\gamma_2$ stands for the standard Gaussian measure.

It is well known (see, for example, the book by Ledoux and Talagrand [24], and papers by Bobkov [4], [5]) that for every measurable $Q \subset \mathbb{R}^n$ such that $\gamma_2(Q) \geq \frac{1}{2}$,

$$\alpha_Q(h) \leq \frac{1}{2} e^{-\frac{h^2}{2}}.$$  

Moreover,

$$\gamma_2(Q + hB^n_2) \geq \gamma_2(H_Q + hB^n_2),$$

where $H_Q$ is a half space such that $\gamma_2(Q) = \gamma_2(H_Q)$ (Theorem 1.2 in [24]).
I have investigated the relation of the estimates for \( \alpha_Q(h) \) with the Gaussian surface area \( \gamma_2(\partial Q) \). For some sets \( Q \) it allows to improve the inequality \( \square \) for certain range of \( h \). Namely, in \( \square \) I prove the following

**Theorem 4.1.** For any convex set \( Q \subset \mathbb{R}^n \) containing the origin and for any \( h \in [0, \frac{4\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)}] \),

\[
\gamma_2(Q + hB_2^n) \geq \gamma_2(Q) + \frac{\sqrt{\pi}\gamma_2(\partial Q)^2}{8\sqrt{n}} \cdot \left(1 - e^{-\frac{\sqrt{\pi}}{\sqrt{\pi}\gamma_2(\partial Q)} h}\right).
\]

Let \( Q \) be a convex set in \( \mathbb{R}^n \) such that \( \gamma_2(Q) \geq \frac{1}{2} \) and \( \gamma_2(\partial Q) \geq \frac{8}{\sqrt{2\pi}} \). Then the estimate given by Theorem 4.1 is stronger then \( \square \) for all \( h \in [0, e^{\frac{\gamma_2(\partial Q)\log\gamma_2(\partial Q)}{\sqrt{n}}} \] , where \( c \) is an absolute constant.

4.1. **Current work and open problems.** A number of results in many areas of mathematics heavily rely on the inequality \( \square \) and results of its type (see, for example, the work of Rudelson and Vershynin \( \square \)). I am hoping to obtain results using Theorem 4.1 as a tool.

I am also working on generalizations of Theorem 4.1 to wider classes of measures.

5. **On the Gaussian Brunn–Minkowski inequality.**

Consider the standard Gaussian measure \( \gamma_2 \). Gardner and Zvavitch \( \square \) had studied the inequality \( \square \), that is

\[
\gamma_2(\lambda A + (1 - \lambda)B)^{\frac{1}{n}} \geq \lambda^{\gamma_2(A)^{\frac{1}{n}}} + (1 - \lambda)\gamma_2(B)^{\frac{1}{n}},
\]

where \( A \) and \( B \) are convex sets in \( \mathbb{R}^n \). Although \( \square \) is false in its full generality, there is a hope that it is true under some natural assumptions on \( A \) and \( B \). I have shown that the inequality \( \square \) holds true for \( A, B \subset \mathbb{R}^2 \) when both \( A \) and \( B \) are close to a disc. The approach I use to attack this problem was inspired by the work of Colesanti, Hug, Saorín-Gomez \( \square \), \( \square \).

The support function \( h_Q \) of a convex set \( Q \subset \mathbb{R}^n \) is the function on the unit sphere defined by \( h_Q(\theta) = \max_{x \in Q} \langle x, \theta \rangle \). By homogeneity it extends from the sphere to the whole space.

Let \( h(u) \) be a strictly convex \( C^2 \)-smooth function on the sphere \( S^{n-1} \). Consider a function \( \psi(u) \in C^2(S^{n-1}) \). Pick a (small enough) positive number \( a \). For each \( s \in [0, a] \), consider the function \( h_s = h + s\psi \). By \( K_s \) denote the convex set with the support function \( h_s(u) \). Introduce the notation for the family of sets \( K_s \):

\[
K_n(h(u), \psi(u), a) := \{K_s\}_{s \in [0, a]}.
\]
For a given positive number \( R \), the notation \( K_n(R, \psi(u), a) \) stands for the family \( K_n(h(u), \psi(u), a) \) when \( h(u) \equiv R \).

I have shown the following Proposition.

**Proposition 5.1.** Pick \( R \in (0, \infty) \). Fix \( \psi \in C^2(S^1) \). Then there exists an \( \epsilon = \epsilon(R, \psi) \) such that for every \( K, L \in K_2(R, \psi, \epsilon) \) (see (7) for the notation), and for every \( \lambda \in [0, 1] \),

\[
\gamma_2^\frac{1}{2}(\lambda K + (1 - \lambda)L) \geq \lambda \gamma_2^\frac{1}{2}(K) + (1 - \lambda) \gamma_2^\frac{1}{2}(L).
\]

I have obtained the formula expressing a measure of a set in terms of its support function. That formula is the key ingredient of my approach to study the inequality (1). The formula is stated in the next Proposition.

**Proposition 5.2.** Let \( \gamma \) be a measure in \( \mathbb{R}^n \) with density \( f \). Let \( K \) be a strictly convex body in \( \mathbb{R}^n \) containing the origin with the support function \( h(u) \in C^2(S^{n-1}) \). By \( \det Q(h(u)) \) denote the curvature function of \( K \). Denote the gradient of \( h \) by \( \nabla h \). Then

\[
\gamma(K) = \int_{S^{n-1}} \frac{h(u) \det Q(h(u))}{|\nabla h(u)|^n} \int_0^{|
abla h|} t^{n-1} f \left( t \cdot \frac{\nabla h}{|\nabla h|} \right) dt du.
\]

For the standard Gaussian measure on the plane this formula simplifies.

**Corollary 5.3.** Let \( \gamma_2 \) be the Standard Gaussian measure in \( \mathbb{R}^2 \). Let \( K \) be a strictly convex body in \( \mathbb{R}^2 \) containing the origin with the support function \( h(u) \in C^2(S^1) \). Then

\[
\gamma_2(K) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h^2(u) + \dddot{h}(u)}{h^2 + \dddot{h}^2} \left( 1 - e^{-\frac{\dddot{h}^2}{2}} \right) du.
\]

Corollary 5.3 can be rewritten in even simpler way: under the assumptions of Corollary 5.3

\[
\gamma_2(K) = \frac{1}{4\pi} \int_{-\pi}^{\pi} (h^2 + \dddot{h}) \int_0^1 e^{-t\frac{\dddot{h}^2}{2}} dt du.
\]

The conjecture of Gardner and Zvavitch [16] suggests that the inequality (1) is true when both of the sets \( A \) and \( B \) are origin-symmetric. I am working on this conjecture. Jointly with Nazarov, I have obtained a criterion of the inequality (1) in terms of the support function:

**Proposition 5.4.** The inequality (1) holds for every pair of convex symmetric sets \( A \) and \( B \) in \( \mathbb{R}^2 \) if and only if for any even functions \( h(u) \in C^2[-\pi, \pi] \) and \( \psi(u) \in C^2[-\pi, \pi] \) such that \( h(-\pi) = h(\pi) \), \( \dot{h}(-\pi) = \dot{h}(\pi) \), \( \dddot{h}(-\pi) = \dddot{h}(\pi) \), \( \psi(-\pi) = \psi(\pi) \), \( \dot{\psi}(-\pi) = \dot{\psi}(\pi) \) and
\( \tilde{\psi}(-\pi) = \tilde{\psi}(\pi), \) and also \( h(u) + \tilde{h}(u) > 0 \) for all \( u \in [-\pi, \pi], \) the following inequality holds:

\[
2 \cdot \left[ \int_{-\pi}^\pi \left( \psi^2(1-h(h+\tilde{h})) - \dot{\psi}^2 \right) e^{-\frac{h^2+\dot{h}^2}{2}} \, du \right].
\]

\[
\left( \int_{-\pi}^\pi \dot{\psi}(h+\tilde{h}) e^{-\frac{h^2+\dot{h}^2}{2}} \, du \right)^2.
\]

For polytopes the inequality \((5)\) involves a finite set of parameters. I am checking whether \((5)\) is true for symmetric polytopes using a computer program; the data I am getting suggests that \((5)\), and, consequently, \((1)\) are correct for all symmetric sets.

### 5.1. Current work and open problems.

The proof of Proposition \(5.1\) involves deriving \((5)\) for the case \( h = const \) from Poincare inequality. I am hoping to find the proof of \((5)\) for even functions \( h \) and \( \psi \) in the full generality. My intuition is that it should follow from Poincare inequality as well. My ideas on how to attack this problem include deriving \((5)\) from the B-Theorem of Cordero-Erausquin, Fradelizi and Maurey \([11]\), restated in the appropriate way.

In addition to the previously stated results, I have proved the equivalent of Proposition \(5.1\) for a wider class of measures including the standard Gaussian measure. I am also working on extending Proposition \(5.1\) to the \( n \)-dimensional case.

I believe that Proposition \(5.2\) and Corollary \(5.3\) are of independent interest and may find applications in questions related to B-Theorem \([11]\), S-Theorem of Latala and Oleszkiewicz \([23]\) and isoperimetric type questions.

### References


[25] G. V. Livshyts, *Maximal surface area of a convex set in $\mathbb{R}^n$ with respect to exponential rotation invariant measures*, Journal of Mathematical Analysis


