1. Using either the methods of complex variables or of real analysis prove that the following improper Riemann integral exists:

\[ \int_0^\infty \frac{\sin(x)}{x} \, dx. \]

2. Define the Lebesgue outer regular measure \( m^* \) of a subset \( E \) of \((0,1)\) by

\[ m^*(E) = \inf \{ m^*(U) \mid E \subset U, U \text{ open} \}. \]

Define the Lebesgue inner regular measure of \( E \) to be

\[ m_*(E) = \sup \{ m^*(K) \mid K \subset E, K \text{ closed} \}. \]

Prove the following two definitions of measurability are equivalent.

**Definition 1:** We say \( E \) is measurable iff

\[ m^*(E) + m^*(E') = 1 \]

where \( E' \) denotes the complement of \( E \) in \((0,1)\).

**Definition 2:** We say \( E \) is measurable iff

\[ m^*(E) = m_*(E). \]

3. Let \( X \) and \( Y \) be two measure spaces.
   (a) Give the definition of a measurable function from \( X \) to \( Y \).
   (b) Let \( f \) and \( g \) be measurable functions from \( X \) to \( Y \). Prove that \( f + g \) is measurable.
   (c) Let \( f \) and \( g \) be measurable functions from \( X \) to \( Y \). Prove that \( fg \) is measurable.

4. Let \((X, \mathcal{M}, \mu)\) be a measure space and let \( \{f_n\} \) and \( \{g_n\} \) be two sequences of non-negative measurable functions on \( X \). Assume \( f_n \leq g_n \) for all \( n \) and
   (i) \( f_n \to f \) pointwise almost everywhere.
   (ii) \( g_n \to g \) pointwise almost everywhere.
   (iii) \( g \) is integrable and \( \int_X g_n \, d\mu \to \int_X g \, d\mu \).

Prove that \( f \) is integrable and \( \int_X f_n \, d\mu \to \int_X f \, d\mu \)

5. Let \( X \) be a set and let \( \mathcal{M} \) be a sigma algebra of subsets of \( X \). Suppose that \((X, \mathcal{M}, \beta)\) and \((X, \mathcal{M}, \nu)\) are two finite measure spaces. We say that \( \beta \) is absolutely continuous with respect to \( \nu \) \( (\beta \ll \nu) \) if \( \beta(E) = 0 \) whenever \( \nu(E) = 0 \). Prove that \( \beta \ll \nu \) if and only if for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( \nu(E) < \delta \) implies that \( \beta(E) < \epsilon \).
6. Given three vertices of a parallelogram \( z_1, z_2, z_3 \) in \( \mathbb{C} \), find the fourth vertex \( z_4 \), opposite to the vertex \( z_2 \), in terms of the other three vertices.

7. Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be an entire function. Calculate each of the following integrals, in terms of the function \( f \) and its derivatives. (Here, \( C_s(b) = \{ z \in \mathbb{C} : |z - b| = s \} \).)

   - (a). \( \frac{1}{2\pi i} \int_{C_s(a)} \frac{f^2(z)}{(z - a)^2} \, dz \),

   - (b). \( \left[ \frac{1}{2\pi i} \int_{C_s(a)} \frac{f(z)}{z - a} \, dz \right]^2 \),

   - (c). \( \frac{1}{2\pi i} \int_{C_s(a)} \frac{f(z)}{(z - a)^2} \, dz \)

   - (d). \( \frac{1}{2\pi i} \int_{C_s(a)} \frac{f^2(z)}{z - a} \, dz \).

8. Let \( f : D \rightarrow D \) be an analytic function, where \( D \) is the open unit disc in \( \mathbb{C} \). Suppose that \( f(1/4) = -2/3 \). Is it possible for \( f(1/3) = 2/3 \)? Explain your reasoning.

9. (a). State the Cauchy-Riemann equations and prove that if \( f \) is differentiable at a point \( b \in \mathbb{C} \), then the Cauchy-Riemann equations hold at \( b \).

   (b). Let \( f : U \rightarrow \mathbb{C} \) be an analytic function on the domain \( U \) such that \( \text{Re} f(z) = (\text{Im} f(z))^2 \) for all \( z \in U \). Prove that \( f \) is a constant function.

10. Find the domains of convergence of the given series.

    - (a). \( \sum_{n=0}^{\infty} \left( z^n + \frac{1}{2^n} z^n \right) \),

    - (b). \( \sum_{n=0}^{\infty} \frac{(-1)^n}{z + n} \).