ANSWER ALL QUESTIONS

1. Let \( f_n(z), n = 0, 1, 2, \ldots \) be holomorphic on a region \( D \). Let \( f(z) = \sum_{n=0}^{\infty} f_n(z) \). Show that if the series is uniformly convergent on every compact subset of \( D \), then \( f(z) \) is holomorphic on \( D \).

2. How many roots (counting multiplicities) does
\[
z^6 - 5z^2 + 10 = 0
\]
have in the annulus \( \Omega := \{ z : 1 < |z| < 2 \} \)?

3. Find the Laurent expansion of \( \frac{z^2 + z + 1}{z(z + 1)^2} \), in powers of \((z - 1)\), that is convergent for \( 1 < |z - 1| < 2 \).

4. a) Let \( f(z) \) be an entire function such that \( |f(z)| \to \infty \) as \( |z| \to \infty \). Prove that \( f(z) \) is a polynomial.

b) Show that if \( f(z) = u + iv \) is an entire function such that \( u_x + v_y = 0 \), then \( f(z) = az + b \) where \( a \) and \( b \) are complex constants and \( \text{Re}(a) = 0 \).
5. Use contour integration to evaluate \( \int_{0}^{2\pi} \frac{\sin^2 \theta d\theta}{5 + 4 \cos \theta} \).

6. Let \( f(z) \) be a meromorphic function satisfying
\[
|f(z)| \leq \frac{C}{|z - 1|^2} + \frac{C}{|z - 2|^2}.
\]
Prove that \( f(z) = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-2} \) for some complex constants \( A, B, C \).

7. Under the transformation \( f(z) = \frac{z^2}{z-1} \), identify the image of

a) the half plane \( \{z : \text{Re}z > 0\} \),

b) the half plane \( \{z : \text{Im}z > 0\} \),

c) the closed unit disk \( \{z : |z| \leq 1\} \).

8. Let \( f(z) \) be holomorphic with \( f'(z) \neq 0 \) in a region \( D \). Let \( z_0 \in D \) and assume that \( f(z_0) \neq 0 \). Given \( \epsilon > 0 \), show that there exists a \( z \in D \) and a \( \zeta \in D \) such that \( |z - z_0| < \epsilon, |\zeta - z_0| < \epsilon \), and \( |f(z)| > |f(z_0)|, |f(\zeta)| < |f(z_0)| \).

9. An entire function \( f(z) \) satisfies \( f(0) = 1 \) and \( |f(z)| \leq me^{|z|} \) for all complex numbers \( z = x + iy \), where \( m > 0 \) is a constant. Prove that \( f(z) = e^z \) for every \( z \). Could the same inference be made if the estimate on \( f \) were weakened to \( |f(z)| \leq me^{\frac{|z|}{2}} \)?

10. For each positive integer \( k \), \( \{z_{k,j}\}_{j=1}^{k} \) are \( k \) given points which satisfy \( |z_{k,j}| = k^2 \). What is the most general entire function \( f(z) \) which has \( \{z_{k,j}\}_{j=1,k=1}^{k,\infty} \) as its only zeros?