1. Evaluate the following integrals.

   a) \[ \oint_{|z|=2} z^2 e^{1/z} \, dz. \]

   b) \[ \oint_{|z|=2} \frac{\cos z}{z^2(z - 1)} \, dz. \]

2. Find all possible Laurent expansions of \( f(z) = \frac{1}{z^2 - 1} \) about \( z = 1 \).

3. Let \( f \) and \( g \) be entire functions which satisfy

   (i) \( f(0) = g(0) \neq 0 \) and 
   (ii) \( |f(z)| \leq |g(z)| \)

   for every complex number \( z \). Prove that \( f = g \).

4. Let \( C_1 \) be the circle with center 0 and radius 1, and let \( C_2 \) be the circle with center \( \frac{1}{2} \) and radius \( \frac{1}{2} \).

   Consider the function \( f(z) = (z - 1)^{-1} \).

   a) Determine the image under \( f \) of the region between \( C_1 \) and \( C_2 \).

   b) Determine the image under \( f \) of the region in the first quadrant between \( C_1 \) and \( C_2 \).
5. Determine all entire functions $f(z)$ which have the property that there exists a real number $M$ such that $\text{Re}\{f(z)\} - x \leq M$ for all $z = x + iy \in \mathbb{C}$.

6. Let $|a_m| < 1$, $m = 1, 2, \ldots, n$ and

$$F(z) = \prod_{m=1}^{n} \left[ \frac{z - a_m}{1 - \bar{a}_m z} \right].$$

Prove that if $|b| < 1$, the equation $F(z) = b$ has exactly $n$ roots in the open unit disk.

7. Let $f$ be analytic on an open set containing $\bar{\Delta} := \{z \in \mathbb{C} : |z| \leq 1\}$.
   a) Show that if $|f(z)| < 1$ for every $|z| = 1$ then there is a unique $z_0 \in \Delta$ with $f(z_0) = z_0$. (Note: $\Delta = \{z \in \mathbb{C} : |z| < 1\}$).
   b) Give an example to show that this conclusion no longer holds if we relax the condition on $f$ to "$|f(z)| \leq 1$ for every $|z| = 1$".
   c) Show that if $|f(z)| > 1$ for every $|z| = 1$ and $f(0) = 1$ then $f$ must have a zero in $\Delta$.

8. Let $C$ be the unit circle, oriented counter clockwise. For any $z$ in the complex plane, $|z| \neq 1$, evaluate

$$\int_{C} \frac{\bar{z} d\zeta}{\zeta - z}.$$ 

9. Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and suppose that $f : \Delta \to \mathbb{C}$ is an analytic function which is also one-to-one. Assume that $f(0) = 0$, $f'(0) = 1$, and $f$ is not the identity map on $\Delta$. Prove that
   a) $f(\Delta) \not\subset \Delta$,
   b) $\Delta \not\subset f(\Delta)$.
10. Let \( g(z) = \cos(\sqrt{z}) := \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(2k)!} \).

a) Prove that \( g(z) \) is an entire function of \( z \).

b) Find an infinite product representation of \( g(z) \).