1. (Rounding-Error Analysis) Consider the basic arithmetic operations 
\( z_1 := x + y, \)
\( z_2 := x \cdot y, \) and \( z_3 := x / y \) realized in finite-precision/floating-point arithmetic (where \( x \) and \( y \) are not necessarily machine numbers).
   
   (a) Using simple error analysis, derive expressions (to leading order) for the relative error in each.
   
   (b) Show that \( |(\text{fl}(z_i) - z_i)/z_i| \leq 3 \epsilon_p \) for \( i = 2, 3. \)
   
   (c) The same is not true for \( z_1. \) Is the problem mathematical conditioning or numerical conditioning (also called numerical stability)? Explain.

2. (Conditioning) The ordinary dot product of two real \( n \)-vectors \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) is given by
\[
a \cdot b = a_1 b_1 + \cdots + a_n b_n.
\]
This can be viewed as a function on \( \mathbb{R}^{2n} \) to \( \mathbb{R}^n \):
\[
f(a, b) = f(a_1, \ldots, a_n, b_1, \ldots, b_n) := a_1 b_1 + \cdots + a_n b_n.
\]
Assume \( \mathbb{R}^{2n} \) is normed by the usual Euclidean norm
\[
x = (x_1, \ldots, x_{2n}) \implies \|x\|_2 := \sqrt{x_1^2 + \cdots + x_{2n}^2}
\]
and \( \mathbb{R}^n \) is normed by the usual absolute value function.

(a) Derive the following expression for the mathematical condition number of this function:
\[
\text{cond}(f; a, b) = \frac{a_1^2 + \cdots + a_n^2 + b_1^2 + \cdots + b_n^2}{|a_1 b_1 + \cdots + a_n b_n|}.
\]
(HINT: you may use the fact that for \( x, y \in \mathbb{R}^n, \)
\[
\sup_{y \neq 0} \frac{|x \cdot y|}{\|y\|_2} = \|x\|_2.
\]

(b) What situations here are well/ill-conditioned?
3. (Interpolation) Let $P_{i_0 \cdots i_k}$ denote the polynomial of degree $k$ that interpolates to $f(x)$ at $x_{i_0}, \ldots, x_{i_k}$.

(a) Prove that

$$P_{i_0 \cdots i_{k+1}}(x) = \frac{(x - x_{i_0})P_{i_1 \cdots i_{k+1}}(x) + (x_{i_{k+1}} - x)P_{i_0 \cdots i_k}(x)}{(x_{i_{k+1}} - x_{i_0})}.$$  

(b) What is the name of the algorithm that recursively uses this relationship to evaluate the interpolating polynomial (without explicitly forming it) at a given point $x$?

4. (Trigonometric Interpolation) Let $0 < x_0 < x_1 < \ldots < x_{2n} < 2\pi$ and let $\{y_k\}_{k=0}^{2n}$ be real numbers. Prove that there exists a unique trigonometric polynomial

$$T(x) = \frac{1}{2}a_0 + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx)$$

with

$$T(x_k) = y_k \text{ for } k = 0, 1, \ldots, 2n.$$  

5. (Piecewise Polynomial Interpolation) A function $L_\Delta$ is in the class $L_\Delta$ of “real piecewise-linear polynomials” (relative to a given partition $\Delta : a = x_0 < x_1 < \cdots < x_n = b$) if it satisfies

(a) $L_\Delta \in C[a, b]$,

(b) $L_\Delta|_{[x_i, x_{i+1}]} \in \Pi_1$, $i = 0, \ldots, n - 1$,

that is, $L_\Delta$ is a continuous function that coincides with a polynomial of degree at most one in each subinterval.

(a) The classical interpolation problem for $L_\Delta$ is “given $(x_i, f_i)$, $i = 0, \ldots, n$, find $L_\Delta \in L_\Delta$ such that $L_\Delta(x_i) = f_i$, $i = 0, \ldots, n$.”

Prove that this problem has a unique solution.

(b) Supposing that the data come from a twice continuously differentiable function, i.e., $f_i = f(x_i)$, $i = 0, \ldots, n$, ($f \in C^2[a, b]$), derive the error bound

$$\|f - L_\Delta(f)\|_\infty \leq \frac{1}{8} \|f''\|_\infty \|\Delta\|^2,$$

$$\|\Delta\| := \max_{j=0,\ldots,n-1} |x_{j+1} - x_j|.$$
6. (Quadrature, Peano Kernels)

(a) Derive the Peano kernel for the error functional in the Mid-Point Rule:

\[ \int_0^h f(x) \, dx \approx f(h/2) \cdot h. \]

(b) Sketch \( K(t) \).

(c) Derive from the Peano-kernel representation error formulas/bounds in terms of \( f''(\xi) \), \( \| f'' \|_\infty \), and \( \| f'' \|_1 \).

7. (Extrapolation/Romberg Integration)

(a) Granted the validity of the asymptotic expansion for the composite Trapezoid rule

\[ T(h) \sim \int_a^b f(x) \, dx + \tau_1 h^2 + \tau_2 h^4 + \cdots, \]

where

\[ T(h) := h \left[ \frac{f(a)}{2} + f(a + h) + \cdots + f(b - h) + \frac{f(b)}{2} \right], \]

one can inductively define higher-order asymptotic expansions \( T_k(h) \) via

\[ T_0(h) := T(h) \]
\[ T_{k+1}(h) := T_k(h) + \frac{T_k(h) - T_k(2h)}{4^{k+1} - 1}, \quad k = 0, 1, \ldots. \]

An alternate approach is to use polynomial interpolation. As an illustration, find the polynomial of the form \( P_2(x) = c_0 + c_2 x^2 \) that interpolates to the data \((h, T(h))\) and \((2h, T(2h))\). Show that \( P_2(0) = T_1(h) \).

(b) Show also that \( T_1(h) \) is identical with the composite Simpson’s rule.

8. (Direct Solution of Linear Systems)

(a) Gauss elimination with partial pivoting produces matrices \( P \), \( L \), and \( R \), where \( P = P_{n-1} \cdots P_1 \) (with the elementary permutation matrix \( P_j \) interchanging rows \( j \) and \( r_j \)), \( L \) is unit lower triangular, and \( R \) is upper triangular. Roughly how many flops (floating-point operations) are required to compute this decomposition?

(b) Construct a detailed algorithm (in pseudo-code) that uses \( P \), \( L \), and \( R \), as given above, to solve \( Ax = b \).

(c) Roughly how many flops does this algorithm require?

9. (Numerical Linear Algebra)

(a) Let \( \| \cdot \| \) be a norm on \( \mathbb{C}^n \). Define the matrix norm \( \| \cdot \|^* \) induced by \( \| \cdot \| \).

(b) Prove that the induced matrix norm \( \| \cdot \|^* \) is consistent with the underlying vector norm \( \| \cdot \| \).

(c) Give computational formulas for the following vector and matrix norms: \( \| x \|_1 \), \( \| x \|_2 \), \( \| x \|_\infty \), \( \| A \|_1 \), \( \| A \|_F \), and \( \| A \|_\infty \).
10. (Orthogonal Triangularization/Householder Transformations) Let \( v \) be a real non-trivial \( n \)-vector. The \( n \times n \) matrix
\[
P := I - \frac{2vv^T}{v^Tv}
\]
is called a Householder transformation.

(a) If \( Px = ke_1 \), where \( e_1 = (1, 0, \ldots, 0)^T \) and \( k \) is a constant, then show that \( v \) must be in \( \text{span}\{x, e_1\} \).

(b) Determine a vector \( v \) such that \( Px = ke_1 \) for some constant \( k \). (HINT: seek \( v \) in the form \( v = x + \alpha e_1 \)).

11. (Gershgorin Circle Theorem) Let \( A = (a_{i,j}) \) be a complex \( n \times n \) matrix. Prove that all eigenvalues of \( A \) lie in
\[
\bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{i,i}| \leq \sum_{\substack{j=1 \atop j \neq i}}^{n} |a_{i,j}| \right\}.
\]

12. (Iterative Methods)

(a) Define what it means for a real \( n \times n \) matrix \( A \) to be strictly diagonally dominant.

(b) Describe the Jacobi iterative method to solve the linear system \( Ax = b \).

(c) Prove that if \( A \) is strictly diagonally dominant, then the Jacobi iterative method will converge for any starting vector \( x_0 \).