INSTRUCTIONS: Do any 10 of the following 12 problems.

1. (Floating-Point Arithmetic)
   (a) Give the definition of a normalized, $t$-digit, base-$B$, floating-point number (define all terms, notation, component parts).
   (b) Prove that
   \[ |x - \text{rd}(x)| \leq \frac{1}{2} B^{1-t} |x| , \]
   where \( \text{rd}(\cdot) \) denotes the “natural rounding function.”

2. (Conditioning) Consider the function
   \[ f(x_1, \ldots, x_n) = x_1 + \cdots + x_n \]
as a function on \( \mathbb{R}^n \) to \( \mathbb{R} \), with \( \mathbb{R}^n \) normed by the infinity norm and \( \mathbb{R} \) normed by the usual absolute value function.
   (a) Derive the following expression for the mathematical condition number of this function:
   \[ \text{cond}(f; x) = \frac{n \cdot \max\{|x_1|, \ldots, |x_n|\}}{|x_1 + \cdots + x_n|} . \]
   (b) What situations here are well/ill conditioned?

3. (Chebyshev Polynomials/Interpolation)
   (a) Show that
   \[ T_n(x) = \cos(n \arccos(x)), \quad x \in [-1, 1] \]
is a polynomial of degree \( n \) with extrema at
   \[ x_k = \cos\left(\frac{k\pi}{n}\right), \quad k = 0, 1, \ldots, n \]
   and leading coefficient \( 2^{n-1} \). (Hint: \( \cos((n+1)\theta) + \cos((n-1)\theta) = 2 \cos \theta \cos n\theta \))
   (b) Use (a) to show that if \( f \in C^{n+1}[-1, 1] \) and if \( P(x) \) is the polynomial with degree at most \( n \) that interpolates to \( f \) at \( x_k, \quad k = 0, 1, \ldots, n \), then
   \[ \|f(x) - P(x)\|_\infty \leq \frac{1}{2^{n-1}(n+1)!} \|f^{(n+1)}\|_\infty . \]
4. (Generalized Interpolation)

(a) Divided differences can be viewed as linear functionals on \( \Pi_n \). For fixed, distinct \( x_0, \ldots, x_n \), define

\[
\lambda_k(f) := f[x_0, \ldots, x_k], \quad k = 0, \ldots, n,
\]
\[
\phi_0(x) := 1,
\]
\[
\phi_k(x) := (x - x_0) \cdots (x - x_{k-1}), \quad k = 1, \ldots, n.
\]

Show that \( \{\lambda_k\} \) and \( \{\phi_k\} \) are dual bases in the sense that \( \lambda_k(\phi_l) = \delta_{kl} \). It follows (given that the \( \phi_k \) span \( \Pi_n \)) that

\[
P \in \Pi_n \Rightarrow P = \sum_{k=0}^{n} \lambda_k(P)\phi_k.
\]

(b) Another basis for \( \Pi_n \) is the Lagrange basis \( \{L_i\}_{i=0}^{n} \). What is a dual basis of linear functionals associated with these?

(c) Another basis of linear functionals in \( \Pi_n \) is given by \( \mu_k(f) := f^{(k)}(x_0), \quad k = 0, \ldots, n \). What is the polynomial basis that is dual to these?

(d) Let \( \{\Lambda_k\}_{k=0}^{n} \) be \( n + 1 \) linear functionals on \( \Pi_n \). When does the generalized interpolation problem of finding \( P \in \Pi_n \) such that

\[
\Lambda_k(P) = f_k, \quad k = 0, \ldots, n
\]

have a unique solution for any given data \( f_0, \ldots, f_n \)?

5. (Quadrature) A quadrature formula

\[
I_n(f) = \sum_{j=0}^{n} \omega_{n,j}f(x_j) \approx \int_a^b f(x) \, dx
\]

is called interpolatory if

\[
\omega_{n,j} = \int_a^b \prod_{\substack{i=0 \atop i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)} \, dx.
\]

The precision of a quadrature method is the greatest integer \( n \) such that

\[
I_n(P) - \int_a^b P(x) \, dx = 0
\]

for all polynomials of degree \( n \). Show that a quadrature method with precision at least \( n \) is interpolatory.
6. (Quadrature/Peano Kernels)

(a) State (DO NOT PROVE) the “Peano Kernel Theorem.”

(b) Consider the composite Trapezoid quadrature rule:

\[
\int_a^b f(x) \, dx \approx h \left[ \frac{1}{2} f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right],
\]

\[x_i = a + ih, \quad i = 0, \ldots, n, \quad h := (b - a)/n.\]

i. What is the largest polynomial space \( \Pi_k \) for which this quadrature rule is exact?

ii. The associated error functional is

\[R(f) := \int_a^b f(x) \, dx - h \left[ \frac{1}{2} f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right].\]

Prove that this is a bounded linear functional on \( C^l[a,b] \) for \( l = 0, 1, 2, \ldots \).

iii. Without explicitly constructing the kernels, indicate what types of Peano kernel representation formulas are possible for \( R \) (depending on the smoothness of \( f \)).

7. (Orthogonal Polynomials/Gauss Quadrature)

(a) Give the inner products with respect to which the following polynomials are orthogonal: Chebyshev, Hermite, Laguerre, and Legendre.

(b) Let \( x_1, \ldots, x_n \) be the roots of the \( n \)-th orthogonal polynomial with respect to the weighted inner product

\[(f,g)_w := \int_a^b f(x)g(x)w(x) \, dx.\]

Let \( h_1, \ldots, h_n \) be the weights in the quadrature rule

\[\int_a^b f(x)w(x) \, dx \approx \sum_{k=1}^n h_k f(x_k)\]

constructed to be exact on \( \Pi_{n-1} \). Prove that this rule is actually exact on \( \Pi_{2n-1} \).

8. (Full-Pivoting Back Solver) Gauss Elimination with full pivoting applied to a given matrix \( A \) produces matrices \( L \) (unit lower triangular) and \( R \) (upper triangular) and index vectors \( r = (r_1, \ldots, r_{n-1}) \) and \( s = (s_1, \ldots, s_{n-1}) \), such that

\[PAQ = LR.\]

Here \( P = P_{n-1} \cdots P_1 \) (where \( P_j \) is an elementary permutation matrix that permutes rows \( j \) and \( r_j \)), and \( Q = Q_1 \cdots Q_{n-1} \) (where \( Q_j \) permutes columns \( j \) and \( s_j \)).

Given the information in this factorization, develop a complete pseudo-code description of an algorithm to use this to solve \( Ax = b \).
9. (Matrix Condition Number) Let $A$ be a nonsingular $n \times n$ matrix.

(a) Define the matrix condition number, $\kappa(A)$.

(b) Prove that

$$\kappa(A) = \|A\| \cdot \max_{\|y\|=1} \|A^{-1}y\|.$$ 

(c) Assume $A$ is an upper triangular matrix. Describe how (b) can be used to obtain a “good” lower bound for $\kappa_{\infty}(A)$ (where $\kappa_{\infty}(A)$ is the condition number of $A$ with respect to $\| \cdot \|_{\infty}$).

10. (Singular Values, Matrix Norms) Let $A$ be an $m \times n$ real matrix, and let $Q \in \mathbb{R}^{m \times m}$ and $Z \in \mathbb{R}^{n \times n}$ be orthogonal.

(a) Show that $\|QAZ\|_F = \|A\|_F$ and $\|QAZ\|_2 = \|A\|_2$.

(b) Show that $\|A\|_F^2$ equals the sum of the squares of the singular values of $A$ and that $\|A\|_2$ equals the maximum singular value of $A$.

(c) Determine all singular values of $Z$.

11. (Orthogonal Triangularization/Householder Transformations)

(a) The Householder transformation associated with the nontrivial vector $u$ can be written

$$P = I - \beta uu^H, \quad \beta := \frac{2}{u^Hu}.$$ 

i. Prove that $P$ is Hermitian.

ii. Prove that $P$ is unitary.

(b) Given a unitary matrix $P \in \mathbb{C}^{m \times m}$ that “triangularizes” the matrix $A \in \mathbb{C}^{m \times n}$, indicate (briefly) how it can be used to solve the Linear Least Squares problem.

12. (Eigenproblems)

(a) Given a good approximation to an eigenvalue of a matrix $A$, how would you compute an approximate eigenvector associated with it? Describe briefly.

(b) Given an approximation to an eigenvector of $A$, how would you compute an approximate eigenvalue associated with it?

(c) Briefly describe the standard method to compute all the eigenvalues of a Hermitian, tri-diagonal matrix.

(d) Discuss the mathematical conditioning of the eigenvalues for normal, non-defective, and defective matrices.