Numerical Analysis Qualifier

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INSTRUCTIONS: Do any 8 of the following 12 problems.

1. (Polynomial Interpolation)
   (a) Given \( n + 1 \) distinct nodes \( x_0, x_1, \ldots, x_n \) in a real interval \([a, b]\) and associated real numbers \( f_0, f_1, \ldots, f_n \), show that there is a polynomial \( p_n \) of degree at most \( n \), such that
   \[
   p_n(x_i) = f_i, \quad i = 0, 1, \ldots, n. \tag{1}
   \]
   (b) Show that the polynomial \( p_n \) of part (a) is unique.
   (c) Suppose that the numbers \( f_i \) of part (a) are defined by \( f_i = f(x_i) \), \( i = 0, 1, \ldots, n \), where \( f \) is a real-valued function that is differentiable arbitrarily many times on \([a, b]\). Moreover, let there be a constant \( M \), such that
   \[
   \max_{a \leq x \leq b} \left| \frac{d^j}{dx^j} f(x) \right| \leq M
   \]
   for all \( j \geq 0 \). Can it be shown without additional assumptions about the location of the \( x_i \) that \( p_n(x) \) converges uniformly to \( f(x) \) on \([a, b]\) as \( n \to \infty \)? Motivate.

2. (Divided Differences)
   (a) Define divided differences.
   (b) Express the polynomial in Problem 1(a) by Newton’s interpolation formula. How are the coefficients of the polynomials \( p_{n-1}(x) \) and \( p_n(x) \) related? (\( p_k(x) \) is the polynomial that solves the interpolation problem (1) for \( n = k \).)

3. (Discrete Fourier Analysis) Consider the discrete inner product
   \[
   (f, g) = \frac{1}{2N + 1} \sum_{k=-N}^{N} f(x_k)\overline{g(x_k)}, \quad x_k = \frac{2k\pi}{2N + 1},
   \]
   where the bar denotes complex conjugation.
(a) Show that the functions \( f_j(x) = \exp(\sqrt{-1}jx) \), \( j = 0, \pm 1, \pm 2, \ldots, \pm N \),
(where \( i = \sqrt{-1} \)) are orthogonal with respect to this inner product.

(b) Let \( g_k \) be function values associated with the nodes \( x_k \). Describe
a method based on the orthogonality of the \( f_j(x) \) for computing
the trigonometric polynomial
\[
t_N(x) = \sum_{j=-N}^{N} \alpha_j f_j(x),
\]
such that
\[
t_N(x_k) = g_k, \quad k = 0, \pm 1, \pm 2, \ldots, \pm N.
\]
\( N \) is assumed to be a general positive integer.

(c) Assume that \( N = 2^l \) for a positive integer \( l \). What is the “Fast
Fourier Transform” algorithm, and what is the operation count
when applying it to compute the trigonometric polynomial of part
(b) ?

4. (Piecewise Polynomial Interpolation) A function \( L_\Delta \) is in the class \( \mathcal{L}_\Delta \)
of “real piecewise-linear polynomials” (relative to a given partition
\( \Delta : a = x_0 < x_1 < \cdots < x_n = b \) ) if it satisfies
(a) \( L_\Delta \in C[a, b] \),
(b) \( L_\Delta |_{[x_i, x_{i+1}]} \in \Pi_1 \), \( i = 0, \ldots, n - 1 \),
that is, \( L_\Delta \) is a continuous function that coincides with a polynomial
of degree at most one in each subinterval.

(a) Show that \( \mathcal{L}_\Delta \) is a vector space, and determine its dimension.

(b) The classical interpolation problem for \( \mathcal{L}_\Delta \) is “given \((x_i, f_i)\), \( i = 0, \ldots, n \), find \( L_\Delta \in \mathcal{L}_\Delta \) such that
\[
L_\Delta(x_i) = f_i, \quad i = 0, \ldots, n.
\]
Prove that this problem has a unique solution.

(c) Supposing that the data come from a twice continuously differentiable function, i.e., \( f_i = f(x_i) \), \( i = 0, \ldots, n \), ( \( f \in C^2[a, b] \) ),
derive the error bound
\[
\|f - L_\Delta(f)\|_\infty \leq \frac{1}{8} \|f''\|_\infty \|\Delta\|^2, \quad \|\Delta\| := \max_{j=0,\ldots,n-1} |x_{j+1} - x_j|.
\]
5. (Orthogonal Polynomials)
   
   (a) Given the inner product
   \[ (f, g) = \int_a^b f(x)g(x)w(x)dx, \]
   where \( w(x) \) is a weight function, show that the monic, orthogonal polynomials with respect to the inner product \((f, g)\) satisfy a three term recurrence relation. Present this recurrence relation.

   (b) Give \( a, b \) and \( w(x) \) for Legendre and Chebyshev polynomials.

6. (Gaussian Quadrature)
   
   (a) Characterize a Gaussian quadrature rule for the evaluation of \( \int_a^b f(x)dx \). What are the nodes of this quadrature rule?

   (b) What are the weights?

7. (Nonlinear Equations) One wants to solve the equation \( x + \ln x = 0 \), whose root is near \( x = 1/2 \), by iteration, and one chooses between the following iteration formulas
   \[
   \begin{align*}
   x_{n+1} &= -\ln x_n \\
   x_{n+1} &= e^{-x_n} \\
   x_{n+1} &= \frac{x_n + e^{-x_n}}{2}
   \end{align*}
   \]
   
   (a) Which of the formulas can be used? Motivate your answer.

   (b) Which of the formulas should be used? Why?

8. (Gaussian Elimination) A matrix \( H = [h_{ij}]_{i,j=1}^n \in \mathbb{R}^{n \times n} \) is called a "Hessenberg matrix" if \( h_{ij} = 0 \) when \( i > j + 1 \). Let \( b \in \mathbb{R}^n \). How many arithmetic operations are necessary in order to solve
   \[ Hx = b \]
   for \( x \in \mathbb{R}^n \) by Gaussian elimination?

9. (QR Factorization) Let \( H \) be an upper Hessenberg matrix. Describe how it can be factored into
   \[ H = QR \]
   where \( R \) is upper triangular and \( Q \) is orthogonal by
(a) Householder transformations
(b) Givens transformations
(c) Which of these approaches is to be preferred? Motivate.

10. (Iterative Methods) Let $A$ be a nonsingular matrix and consider iterative solutions of the linear system of equations $Ax = b$.

(a) Describe Jacobi, Gauss-Seidel and SOR iteration.
(b) Give sufficient conditions for the Jacobi and Gauss-Seidel methods to converge.
(c) Define “Property A” and “consistently ordered matrices.” How do the concepts relate? What is their significance with respect to the SOR method?

11. (Least Squares Problems) Discuss the mathematical theory of the linear least-squares problem,

$$\min_{x \in \mathbb{R}^n} \|y - Ax\|_2, \quad A \in \mathbb{R}^{m \times n}, \quad y \in \mathbb{R}^m.$$ 

(a) When does this problem possess a solution?
(b) When is the solution unique?
(c) What are the normal equations, and what is their connection with this problem?
(d) Prove that if the columns of $A$ are linearly dependent, then a solution cannot be unique.

12. (Symmetric Tri-Diagonal Eigenproblem) Let $B_i$ denote the $i$-th leading principal submatrix of the tridiagonal matrix

$$B = \begin{bmatrix}
\delta_1 & \bar{\gamma}_2 \\
\gamma_2 & \delta_2 & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\gamma_n & \bar{\gamma}_n & \cdots & \cdots & \delta_n
\end{bmatrix},$$

and let $p_i(\mu) := \det(B_i - \mu I)$.

(a) Derive a 3-term recurrence relation for the polynomials $p_i(\mu)$.
(b) Describe (briefly) the usefulness of these polynomials in solving numerically the eigenproblem for $B$. 

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