INSTRUCTIONS: Do any 8 of the following 10 problems.

1. (Conditioning) Suppose that you need to code up a function subprogram to evaluate (with high relative accuracy) the function

\[ f(x) := \frac{x}{e^{-x} - 1}, \]

for any real \( x \).

(a) What is the mathematical condition number of this function? For what ranges of \( x \) is this evaluation well conditioned, ill conditioned (with respect to relative error)?

(b) Give a mathematically equivalent formula that is better suited for numerical computation when \( x \) is close to 0. Explain why.

2. (Polynomial Interpolation)

(a) Given \( n + 1 \) distinct nodes \( x_0, x_1, \ldots, x_n \) in a real interval \([-1, 1]\) and associated real numbers \( f_0, f_1, \ldots, f_n \), show that there is a polynomial \( p_n \) of degree at most \( n \), such that

\[ p_n(x_i) = f_i, \quad i = 0, 1, \ldots, n. \]

(b) Show that the polynomial \( p_n \) of part (a) is unique.

(c) Suppose that the numbers \( f_i \) of part (a) are defined by \( f_i = f(x_i), i = 0, 1, \ldots, n \), where \( f \) is a real-valued function that is differentiable arbitrarily many times on \([a, b]\). Moreover, let there be a constant \( M \), such that

\[ \max_{a \leq x \leq b} \left| \frac{d^j}{dx^j} f(x) \right| \leq M \]

for all \( j \geq 0 \). Prove or find a counter example to the statement: The polynomials \( p_n(x) \) converges uniformly to \( f(x) \) on \([a, b]\) as \( n \to \infty \) for any distribution of distinct nodes \( x_i \) in \([-1, 1]\).
3. (Chebyshev Equi-Oscillation Theorem) Determine the best uniform linear approximation ($P_1 \in \Pi_1$) to the function $f(x) = e^x$ on $[0,1]$.

4. (Composite Rules)
   (a) Given the basic Trapezoid Rule with error formula
   \[
   \int_a^b f(x) \, dx = \frac{(b-a)}{2} [f(a) + f(b)] - \frac{(b-a)^3}{12} f''(\xi),
   \]
   derive the associated composite Trapezoid Rule with respect to a not-necessarily-uniform partition
   \[
   a = x_0 < x_1 < \cdots < x_n = b, \quad h_{i+1} := x_{i+1} - x_i.
   \]
   (b) Assuming $f \in C^2[a,b]$, derive the following bound on the error:
   \[
   \left| \int_a^b f(x) - T_\Delta(f) \right| \leq \frac{1}{12} h^2 \|f''\|_\infty (b-a), \quad h := \max_{i=1,\ldots,n} h_i,
   \]
   where $T_\Delta(f)$ denotes the composite rule.

5. (Orthogonal Polynomials) Let $d\omega(x)$ be a positive measure on $[a,b]$, and define the inner product
   \[
   (f, g) = \int_a^b f(x)g(x)d\omega(x).
   \]
   (a) Show that the family of monic orthogonal polynomials with respect to the inner product $(\cdot, \cdot)$ satisfy a three term recurrence relation.
   (b) Show that all zeros of any one of the orthogonal polynomials are simple and lie in the interval $[a,b]$.

6. (Generalized Eigenvalue Problems) Consider the generalized symmetric definite eigenvalue problem $Ax = \lambda Bx$, where $A$ and $B$ are real, square, symmetric matrices, and $B$ is positive definite.
   (a) Show that the eigenvalues are real.
   (b) Show that the eigenvectors corresponding to distinct eigenvalues are “$B$-orthogonal” (i.e., $x_i^TBx_i = 0$).
(c) Construct a transformation that allows you to determine the generalized eigenvalues (and eigenvectors) by solving a regular symmetric eigenproblem.

7. (Matrix Representation) Show that any $m \times m$ matrix $A$ admits the representation

$$A = U R U^H,$$

where $U \in \mathbb{C}^{m \times m}$ is a unitary matrix, $R \in \mathbb{C}^{m \times m}$ is upper triangular, and the superscript $H$ denotes transposition and complex conjugation.

8. (Singular Value Decomposition) Let the matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$ be given, and assume that $m \geq n$. Consider the least-squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2.$$

Define the singular value decomposition of $A$, and discuss how it can be used to solve this least squares problem. Do not assume that $A$ is of full column rank.

9. (Rayleigh Quotient) Let $A \in \mathbb{R}^{n \times n}$ be symmetric. The Rayleigh Quotient associated with $A$ is the function

$$r(x) := \frac{x^T Ax}{x^T x}, \quad x \in \mathbb{R}^n.$$

Note that if $(\lambda, x)$ is an eigenpair of $A$, then it follows that $r(x) = \lambda$.

(a) Show that the Rayleigh Quotient is stationary on the eigenvectors of $A$, that is, if $x$ is an eigenvector of $A$, then $\nabla r(x) = 0$ (where $\nabla$ denotes the gradient).

(b) Use this result to show that if $y$ is an approximation to the eigenvector $x$, with associated eigenvalue $\lambda$, then

$$|\lambda - r(y)| = O\left(\|x - y\|_2^2\right).$$

10. (Nonlinear Equation) Let $f \in C^\infty$ have a zero at $x_\ast$.

(a) Show that when the initial approximate solution $x_0$ is sufficiently close to $x_\ast$ and $x_\ast$ is a simple zero, then the iterates $x_k$ determined by Newton’s method converge to $x_\ast$ at least quadratically.

(b) What is the rate of convergence of Newton’s method when $x_\ast$ is a zero of multiplicity 2?