Directions

1. Answer questions completely: a fully-done problem is far more revealing of your understanding than two half-done attempts.

2. State your reasons for your claims. You may cite and use standard results. However, you have to demonstrate that you know why things are so; you cannot assume that we know that you know. That’s the purpose of this examination! An exercise that states a standard result of the theory requires that you supply a proof.
1. (a) State and prove the Borel-Cantelli Lemma and the “zero-one” law.

(b) Prove that if $X_1, X_2, \cdots$ is a sequence of independent and identically distributed random variables then

$$E(|X_1|) < \infty \text{ if and only if } P(|X_n| > n \text{ i.o.}) = 0$$

2. (a) Define convergence in probability, almost everywhere, and in $L_p$ and discuss the relationship between them with illustrations by examples.

(b) Give an example of a sequence $X_1, X_2, \cdots$ so that $E(|X_1|) < \infty$ and $E(X_n) \to 0$ but no subsequences of $\{X_n\}$ converge to zero in probability.

3. (a) Define vague convergence of a sequence of measures and convergence of a sequence of random variables in distribution.

(b) State and prove that each sequence of subprobability measures (s.p.m.) contains a subsequence converging vaguely to a s.p.m.

(c) State the “continuity” theorem.

4. Define the classes $C_K, C_O, C_B, C$. In terms of them state conditions of vague convergence of measures.

5. (a) State a version of the Strong Law of Large Numbers (SLLN).
(b) Let $X_1, X_2, \cdots$ be independent identically distributed random variables so that $P(X_n = i^\alpha) = P(X_n = (-1)^\alpha) = \frac{1}{2}$.
Find all the values of $\alpha$ so that SLLN holds.

6. (a) Discuss relations between convergence in distribution, in probability, and almost surely.

(b) Prove that if $X_n \to X$ in probability then there exists a sequence \{nk\} such that $X_{nk} \to X$ a.e.

7. Let \{Xi\} be a sequence of independent identically distributed random variables with

\[
P(X_1 \leq t) = \begin{cases} 
1 - e^{-\frac{t^2}{2}} & \text{if } t > 0 \\
0 & \text{if } t \leq 0
\end{cases}
\]

Let the random variable $N_n(t)$ be the number of $X_i$ which are $\leq t$. Find $P(N_n(t) = K), K = 0, 1, \cdots, n$.

8. (a) State and prove the Chebyshev inequality.

(b) Suppose $X$ has density $f(x) = \frac{1}{m!}x^m e^{-x}, x \geq 0, m \geq 0$, where $m$ is an integer. Using formula $\int_0^\infty x^m e^{-x} dx = m!,$ show that $P(0 < X < 2(m + 1)) > \frac{m}{m+1}$.

9. (a) State Kronecker’s Lemma.

(b) Let \{Xi\} be independent, identically distributed r.v.p. with $P(X_1 = c^k) = \frac{1}{2^k}; K = 1, 2 \cdots$

For which value of $c$ does \{Xi\} obey the SLLN?
10. (a) State the Kolomogorov-Feller theorem.

(b) State and prove the WLLN in Khintchin's formulation.