Do as many problems as you can. Show your work. All problems carry equal weight.

Problem 1. (i) Explain what it means for a random variable to have a density with respect to the Lebesgue measure.
   (ii) Explain what it means for a pair of random variables to have a joint density with respect to the Lebesgue measure.
   (iii) Let $X$ be a positive random variable with density $f$ with respect to the Lebesgue measure and let $E(X) = \mu < \infty$. Prove or disprove that $g(x, y) := \frac{f(x+y)}{\mu}$ for $x, y \geq 0$ and $g(x, y) := 0$ otherwise, is a joint density of some pair of random variables.
Problem 2. Let $X_1, X_2, \cdots$ be a sequence of iid random variables for which $P(X_1 = 0) = 1 - p$, and $P(X_1 = 1) = p$ for a constant $p \in (0, 1)$. Let $S_n = X_1 + X_2 + \cdots + X_n$ and let $\epsilon > 0$ be fixed. Calculate each of the following limits directly or by appeal to a limit theorem. In the latter case, state the theorem, and in the former case show your calculations.

(i) \[ \lim_{n \to \infty} P \left( \left| \frac{S_n}{n} - p \right| > \epsilon \right) \]

(ii) \[ \lim_{n \to \infty} P \left( \left| \frac{S_n - np}{\sqrt{n}} \right| > \epsilon \right) \]

(iii) \[ P \left( \lim_{n \to \infty} \frac{S_n}{n} = a \right), \quad \text{where } a \text{ is a fixed real number.} \]
Problem 3.  (i) Define the terms conditional probability and conditional expectation relative to a sigma-field.

(ii) Let \((S, \mathcal{F}, P)\) be a probability space and let \(A_1, A_2, \cdots\) be a countable partition of \(S\) generating a sub-sigma field \(\mathcal{G}\) of \(\mathcal{F}\). Discuss the conditional expectation \(E(X|\mathcal{G})\), and give an explicit expression for \(E(X|\mathcal{G})(\omega)\) when \(\omega \in A_1\) (assume that \(P(A_1) > 0\)).
Problem 4. (i) State a form of the central limit theorem.
(ii) Let $X_i$, $i = 1, 2, \ldots$ be an independent identically distributed sequence of random variables, with mean zero and finite (positive) variance. Prove that
\[
\left( \frac{1}{\sqrt{\sum_{i=1}^{n} X_i^2}} \right) \cdot \sum_{i=1}^{n} X_i \xrightarrow{\text{dist}} N(0, 1).
\]
Justify your arguments.
**Problem 5.**  (i) State a form of the weak law of large numbers.  
(ii) State and prove Kolmogorov's inequality.
Problem 6.  (i) Define a submartingale sequence of random variables.
(ii) State the martingale convergence theorem.
(iii) Let \( \{X_n, \mathcal{G}_n\} \) be a submartingale. Prove that \( E(X_A X_n) \leq E(X_A X_{n+m}) \)
for any \( A \in \mathcal{G}_n \) and \( m = 1, 2, \ldots \).
Problem 7. (i) State any convergence result concerning uniformly integrable martingales.
(ii) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables so that $P(X_1 = -1) = P(X_1 = 1) = \frac{1}{2}$. Let $S_n = X_1 + \frac{X_2}{2} + \ldots + \frac{X_n}{n}$. If $\mathcal{G}_k = \sigma(X_1, X_2, \ldots, X_k)$ then prove that $\{S_n, \mathcal{G}_n\}$ is a uniformly integrable martingale.
(iii) Prove or disprove that for the above martingale sequence,
\[
Var \left( \sum_{k=1}^{\infty} \frac{X_k}{k} \right) = \frac{\pi^2}{6}.
\]
Problem 8.  (i) State an inversion formula for characteristic functions.
(ii) State the continuity theorem of characteristic functions.
(iii) Let $X_n \sim \text{Binomial}(n, p_n)$ where $np_n \to 13$ as $n$ gets large. Find the limiting distribution of $X_n$ as $n$ gets large.
Problem 9. (i) State any one of the two Borel-Cantelli lemmas.
(ii) Prove that if $X_1, X_2, \cdots$ is a sequence of independent and identically distributed random variables then $E|X_1| < \infty$ if and only if

$$P(|X_n| > n \text{ i.o.}) = 0$$
Problem 10. Let $X_1, X_2, \ldots$ be a sequence of independent random variables. Let $P(X_i = i) = P(X_i = -i) = \frac{1}{2i^2}$ and $P(X_i = 0) = 1 - \frac{1}{i^2}$. Prove that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

converges to zero in probability. Does this contradict with the Lindeberg's form of the central limit theorem?