1. Let \((X, \| \cdot \|)\) be a complex normed linear space. Show that the norm is a uniformly continuous function from \(X\) to \([0, \infty)\).

2. Let \(\mathcal{H}\) be a Hilbert space with the inner product \(\langle \cdot, \cdot \rangle\) and let \(M\) be a closed linear subspace of \(\mathcal{H}\). Prove for each \(z \in \mathcal{H}\)

\[
\min\{\|x - z\| : x \in M\} = \max\{| \langle z, y \rangle \| y \| = 1\}.
\]

3. A function \(f : [0, 1] \rightarrow \mathbb{R}\) is called Lipschitz continuous if

\[
|f(x) - f(y)| \leq \alpha |x - y| \quad (\ast)
\]

for some \(\alpha > 0\). Let

\[
M(f) = \inf \alpha
\]

where the infimum is taken over all \(\alpha > 0\) for which condition \((\ast)\) holds. For any Lipschitz continuous function \(f\) let

\[
\|f\| = |f(0)| + M(f) \quad (\ast\ast)
\]

Show that the collection of Lipschitz continuous functions is a Banach space with the norm defined by \((\ast\ast)\).

4. Let \(A\) be a measurable subset of \([0, 2\pi]\). Prove

\[
\lim_{n \to \infty} \int_A \cos(nx)\,dx = 0.
\]
5. Prove that the closed the unit ball in $L^2[0, 1]$ is closed in $L^1[0, 1]$.

6. Let $f \in L^1(-\infty, \infty)$. Prove that given $\epsilon > 0$ there is a $\delta > 0$ such that if $E$ is a measurable set with measure less than $\delta$, $\int_E |f|d\mu < \epsilon$.

7. Let $X$ be a normed linear space and let $X^*$ be its dual space. That is, let $X^*$ consist of all linear functionals on $X$. Equip $X^*$ with the norm $\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}$.

Prove that $X^*$ is a Banach space. (In particular, show $X^*$ is complete in this norm.)

8. Let $1 < p < \infty$ and suppose $\frac{1}{p} + \frac{1}{q} = 1$. Assume $g : [0, 1] \to \mathbb{R}$ is a measurable function such that $fg \in L^1([0, 1])$ for every $f \in L^p[0, 1]$. Show that $g \in L^q[0, 1]$.

9. Prove the following partial converse to Jensen’s inequality: If $\phi$ is a function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 \phi(f(x))dx$$

for every bounded measurable function $f : [0, 1] \to \mathbb{R}$ then $\phi$ is convex. **Hint:** Apply the above inequality to the right simple function.

10. (a) Show that in a normed linear space $X$, if $\lim_{n \to \infty} \sum_{k=1}^n x_k$ exists whenever $\sum_{k=1}^\infty \|x_k\| < \infty$, then $X$ is a Banach space.

(b) Using (a), show that $L^1[0, 1]$ is a Banach space.