1. Take as the definition of a Lebesgue measurable function the following: a function $f : [0, 1] \rightarrow (-\infty, \infty)$ is measurable if the pre-image $f^{-1}(I)$ of every interval $I$ is Lebesgue measurable.

Prove that if $f : [0, 1] \rightarrow (-\infty, \infty)$ is measurable, then $f^{-1}(B)$ is Lebesgue measurable for each Borel set $B \subseteq (-\infty, \infty)$.

2. Let $f : [0, 1] \rightarrow (-\infty, \infty)$ be a bounded measurable function. Show that $f$ is the uniform limit of simple measurable functions.

3. Prove, using an $\varepsilon - \delta$ argument, that the function

$$f(t) = \frac{t^2 + 2t + 3}{t - 1}$$

is continuous at $t = 0$.

4. A subset $K$ of $L^1[0,1]$ is uniformly integrable if for each $\varepsilon > 0$ there is an $M_\varepsilon > 0$ so that for all $f \in K$

$$\int_{\{|f(t)| \geq M_\varepsilon\}} |f(t)| dt \leq \varepsilon.$$

Suppose $\Phi : [0, \infty) \rightarrow [0, \infty)$ is an increasing function such that $\lim_{x \to \infty} \frac{\Phi(x)}{x} = +\infty$.

Prove that the set $K'$ of all $f \in L^1[0,1]$ such that $\int \Phi(|f(t)|) dt \leq 1$ is a uniformly integrable set.
5. Let $f$ be a bounded real-valued function defined on the unit square $S = [0, 1]^2$. Assume $f(x, t)$ is a measurable function of $x$ for each $t \in [0, 1]$. Suppose $\frac{df}{dt}$ exists and is bounded on $S$. Show that
\[
\frac{d}{dt} \int_0^1 f(x, t) \, dx = \int_0^1 \frac{df(x, t)}{dt} \, dx.
\]

6. Let $f : (-\infty, \infty) \rightarrow [0, \infty)$ be a measurable function and let $F(t)$ be given by
\[
F(t) = \lambda(\{x : f(x) > t\})
\]
where $\lambda$ is Lebesgue measure, be $f$’s distribution function.
Show that $F : [0, \infty) \rightarrow [0, \infty)$ is non-increasing (hence measurable) and
\[
\int_{-\infty}^\infty f(t) \, dt = \int_0^\infty F(t) \, dt.
\]

7. Prove that there is a constant $C > 0$ so that for each $f \in L^2[0, 2\pi]$,
\[
\|f\|_1 \leq C\|f\|_2.
\]
By means of example show that $L^1[0, 2\pi] \subsetneq L^2[0, 2\pi]$. 
8. Let \( f : [-\pi, \pi] \rightarrow \mathbb{R} \) be given by

\[
f(x) = \begin{cases} 
+1 & \text{if } x \leq 0 \\
-1 & \text{if } x > 0
\end{cases}
\]

(a) Find the Fourier series of \( f \) with respect to the system

\[\{1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots\}\]

(b) Does the above Fourier series converge to \( f \) in \( L^2[-\pi, \pi] \)? Explain.

(c) Does it converge to \( f \) uniformly? Explain.

9. Suppose \( 1 < p < \infty \) and \( f : [0, 1] \rightarrow \mathbb{R} \) is absolutely continuous and \( f' \in L^p[0,1] \). Show that \( f \in Lip(\frac{1}{q}) \) where \( \frac{1}{p} + \frac{1}{q} = 1 \).

\([Lip(\alpha)\) consists of those \( g : [0,1] \rightarrow \mathbb{R} \) for which there is \( K > 0 \) so

\[|g(x) - g(y)| \leq K|x - y|^{\alpha}\]

for all \( x, y \in [0, 1] \).\]