Directions

1. Answer questions completely: a fully-done problem is far more revealing of your understanding than two half-done attempts.

2. State your reasons for your claims. You have to demonstrate that you know why things are so: you cannot assume that we know that you know. That's the purpose of this examination!

3. Five complete solutions will be considered a solid performance. More is, of course, better BUT keep admonition 1) in mind.
1. Let \( f : [0, 1] \to \mathbb{R} \) be defined by \( f(0) = 0 \) and \( f(x) = (-1)^{n+1}n \) for \((n+1)^{-1} < x \leq n\). Prove or disprove each of the following claims:

(a) \( f \in L^1[0, 1] \).

(b) \( f \) is improperly Riemann-integrable in \([0, 1]\).

2. (a) Supposing \((y_n)\) is a sequence of real numbers, define \( \lim n y_n \).

(b) Using the above definition and the Monotone Convergence Theorem derive Fatou's lemma: If \( f_0, f_1, \ldots, f_n, \ldots \in L^1[0, 1] \) are all non-negative functions and if \( f_0(t) = \lim f_n(t) \) for almost all \( t \in [0, 1] \). then

\[
\int_0^1 f_0(t)dt \leq \lim_n \int_0^1 f_n(t)dt.
\]

3. Let \( S(n) = \sum_{1 \leq k \leq n} 1/k \). Evaluate, with justification,

\[
\lim_{n \to \infty} \int_1^n \frac{S(n)}{x^2 \log n} \, dx.
\]

4. (a) Show that in a Banach space \( X \) if \( \sum_n \|x_n\| < \infty \), then \( \lim_n \sum_{k=1}^n x_k \) exists.

(b) Using (a) show that if \( X \) is a Banach space, \( I \) is the identity operator on \( X \) and \( T : X \to X \) is a bounded linear operator with \( \|T\| < 1 \) then there is a bounded linear operator \( S : X \to X \) so that \( (I - T)S = I \). (Hint: long division \( \frac{1}{1-T} \) works!)

5. (a) State the Radon-Nikodym theorem.

(b) Define the measures \( \mu \) and \( \nu \) on \( \mathbb{R} \) by

\[
\mu(E) = \int_E |x| \, dx, \quad \nu(E) = \int_{E \cap [-1, \infty)} x \, dx.
\]

for any measurable set \( E \subseteq \mathbb{R} \). Show that \( \nu \ll \mu \) and compute the Radon-Nikodym derivative \( \frac{d\nu}{d\mu} \).
6. Let $f \in L^1(\mathbb{R})$. Show that $\sum_{n=1}^{\infty} f(nx)$ converges for almost all $x$.

7. In each of the following, circle true or false (no justification required).

- **T** (i) Every absolutely continuous function $f : [0, 1] \to \mathbb{R}$ has bounded variation.
- **F** (ii) Every continuous function $f : [0, 1] \to \mathbb{R}$ of bounded variation is absolutely continuous.
- **T** (iii) If $f : [0, 1] \to \mathbb{R}$ has a derivative that is essentially bounded, then there is a constant $k > 0$ so that $|f(s) - f(t)| \leq k|s - t|$ for all $s, t \in [0, 1]$.
- **F** (iv) If $f : [0, 1] \to \mathbb{R}$ is everywhere differentiable, then $f'$ is measurable in $[0, 1]$.
- **T** (v) If $f : [0, 1] \to \mathbb{R}$ is everywhere differentiable, then $f'$ has point of continuity in $[0, 1]$.
- **F** (vi) For $n \geq 0$, let $f_n \in L^1[0, 1]$ and suppose $\lim_n f_n(t) = f_0(t)$ for almost all $t \in [0, 1]$. Then $(f_n)$ converges to $f_0$ in measure.
- **T** (vii) For $n \geq 0$, let $f_n \in L^1[0, 1]$ and suppose that $(f_n)$ converges to $f_0$ in measure. Then $(f_n)$ converges to $f_0$ almost everywhere.
- **T** (viii) For $n \geq 0$, let $f_n \in L^1(\mathbb{R})$ and suppose that $(f_n)$ converges to $f_0$ almost everywhere. Then $(f_n)$ converges to $f_0$ in measure.
- **T** (ix) For $n \geq 0$, let $f_n \in L^1[0, 1]$ and suppose that $(f_n)$ converges to $f_0$ in measure. Then $(f_n)$ has a subsequence $(g_n)$ that converges to $f_0$ almost everywhere.
- **F** (x) For $n \geq 0$, let $f_n \in L^1(\mathbb{R})$ and suppose that $(f_n)$ converges to $f_0$ in measure. Then $(f_n)$ has a subsequence $(g_n)$ that converges to $f_0$ almost everywhere.

8. Substantiate your conjecture for any of (ii), (v), (vi) or (ix) of #7. [This does NOT commit you to doing #7!]

9. (a) Show that a metric space $X$ is complete if and only if given a decreasing sequence $(F_n)$ of non-empty closed subsets of $X$ for which $\lim_n \text{diam} F_n = 0$, one has $\bigcap_n F_n \neq \emptyset$.

(b) State a corresponding theorem to characterize the compactness of a metric space.
10. Assuming that $0 < a \leq b$, prove that
\[ \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx = \log(b/a). \]

[This is an example of Frullani's integral. One approach is to consider the integral as a
definite integral itself and then justify switching the order of integration.]

11. Let $f : [0, 1] \to \mathbb{R}$ be continuous and suppose that $\int_0^1 x^n f(x) = 0$ for all $n \geq 0$.
Show that $f(x) \equiv 0$, that is, $f$ is identically zero in $[0, 1]$. 
