Directions

1. Answer questions completely: a fully-done problem is far more revealing of your understanding than two half-done attempts.

2. State your reasons for your claims. You may cite and use standard results. However, you have to demonstrate that you know why things are so; you cannot assume that we know that you know. That’s the purpose of this examination! An exercise that states a standard result of the theory requires that you supply a proof.

3. Write your answers on separate blank paper (not the exam). Do not write all the way to the edge, as it will be lost when copied. Thank You!
1. A real valued function \( f \) defined on an interval \([a, b]\) satisfies a Lipschitz condition with constant \( M \) if \( |f(s) - f(t)| \leq M|s - t| \) for all \( s, t \in [a, b] \). Prove that \( f \) satisfies a Lipschitz condition with constant \( M \) if and only if

(i) \( f \) is absolutely continuous on \([a, b]\)

and

(ii) \(|f'(x)| \leq M \) a.e. (relative to Lebesgue measure)

2. Let \( \mu \) be a measure. Suppose \( \{a_n\}_{n=1}^{\infty} \) is a sequence of numbers such that

\[
\sum_{n=1}^{\infty} |a_n| < \infty.
\]

Suppose \( \{f_n\}_{n=1}^{\infty} \) is a sequence of functions in \( L^1(\mu) \) with bounded \( L^1 \) norm; i.e., there exists \( M > 0 \) such that for all \( n \)

\[
\int |f_n| \leq M
\]

Prove that \( g(x) = \sum_{n=1}^{\infty} a_n f_n(x) \) converges a.e. (relative to \( \mu \)), \( g \in L^1(\mu) \), and

\[
\int g d\mu = \sum_{n=1}^{\infty} a_n \int f_n d\mu.
\]

3. A subset \( K \) of \( L^1(0, 1) \) is called uniformly integrable if given \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( m(E) < \delta \) (\( m \) is the Lebesgue measure) then \( \int_E |f| dm < \epsilon \) for all \( f \in K \).

(1) State Holder’s inequality and then using it show that if \( 1 < p < \infty \).

Then the closed unit ball of \( L^p(0, 1) \) is uniformly integrable.

(2) Show that the closed unit ball of \( L^1(0, 1) \) is NOT uniformly integrable, by constructing a suitably ’misbehaved’ sequence in the ball.
4. In each of the following, state whether the given statement is true or false and indicate briefly the reason for your answer:

(a) If \( F : [0, 1] \rightarrow [0, 1] \) is a homeomorphism of \([0, 1]\) onto \([0, 1] \), then \( F \) takes Borel sets to Borel sets.

(b) If \( F : [0, 1] \rightarrow [0, 1] \) is a homeomorphism of \([0, 1]\) onto \([0, 1] \), then \( F \) takes Lebesgue measurable sets onto Lebesgue measurable sets.

(c) If \( F : (0, 1) \rightarrow \mathbb{R} \) is everywhere differentiable, then \( F'(x) \), the derivative of \( F \) at \( x \in (0, 1) \), has some points of continuity.

(d) If \( F : (0, 1) \rightarrow \mathbb{R} \) is differentiable at each point of \((0, 1)\) and \( F'(x_0) = 0 \) and \( F'(x_1) = 1 \) where \( x_0 \) and \( x_1 \) are points in \((0, 1)\). Then there is a point \( x \in (0, 1) \) so that \( F'(x) = \frac{\sqrt{2}}{2} \).

5. Suppose \( 1 \leq p < \infty \), \( f \in L^p(-\infty, \infty) \) and \( g(x) = \int_x^{x+1} f(t)dt \). Prove that \( g \) is continuous and show that \( \lim_{x \to \pm \infty} g(x) = 0 \).

6. If \( f \) and \( g \) are non negative integrable functions on \([0, 1]\) with

\[
\int_0^1 f(x)dx = \int_0^1 g(x)dx = 1,
\]

then the set of points \( x \in [0, 1] \) for which both \( f(x) \) and \( g(x) \) are \( \leq 3 \) has Lebesgue measure \( \geq \frac{1}{3} \).

7. Using Fatou’s lemma, show that if \( (f_n) \) is a bounded sequence in \( L^2 \) that converges almost everywhere to the measurable function \( f \) then \( f \) is in \( L^2 \), too.
8. Let $m^*$ denote Lebesgue outer measure on $\mathbb{R}$. Suppose $A$ and $B$ are two measurable subsets with the property that there are two disjoint open intervals $I$ and $J$ with $A \subset I$ and $B \subset J$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$. You may assume subadditivity of $m^*$.

9. Suppose that $f$ is a non-negative Lebesgue integrable function on $[0, \infty)$ and that $\int_0^\infty f(t)dt = 1$. Evaluate

$$\int_0^\infty \left( \frac{1}{x} \int_x^\infty f(t)dt \right) dx.$$ 

10. Suppose $X$ is a set, $\mathcal{M}$ a sigma-algebra of subsets of $X$, and $\mu, \nu,$ and $\eta$ are (positive, sigma-finite) measures on $\mathcal{M}$ with $\mu \ll \nu \ll \eta$. Show that $\mu \ll \eta$ and

$$\frac{d\mu}{d\eta} = \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\eta}.$$