Qualifying Examination; Statistics; August 24, 2001

Name

Do all problems, presenting clear, complete and well-written solutions. Two partially done problems do not equal one completely solved problems. Direct any questions concerning possible misprints to the proctor. Show all your work. If you need any percentile values of any distribution, please ask the proctor. GOOD LUCK.

1. Let $X_1, \ldots, X_n$ denote a random sample from the uniform distribution on the interval $(0, \theta]$. Two estimates of $\theta$ are

$$\hat{\theta}_1 = 2 \bar{X} \quad \text{and} \quad \hat{\theta}_2 = \frac{n+1}{n} X_{(n)},$$

where $X_{(n)} = \max \{ X_1, \ldots, X_n \}$.

(i) Find the mean of each of the two estimates and identify the unbiased one(s).
(ii) Find the variance of each of the two estimates and identify the one having smaller variance.
(iii) Does there exist a uniformly minimum variance unbiased (UMVU) estimator? If so provide it and justify why it is a UMVU.

2. (i) Define consistency of estimators.
(ii) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having fixed variance $\sigma^2$. Show that

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \left( \frac{n}{n-1} \right) \left( \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right)^2 \right)$$

is consistent for $\sigma^2$.

3. (i) Define sufficiency of a statistic.
(ii) Let $X_1, \ldots, X_n$ be a random sample from a distribution that has density

$$f(x) = \frac{1}{\Theta} e^{-\frac{x}{\Theta}}, \text{ for } 0 \leq X < \infty.$$ Show that $\bar{X}$ is sufficient for $\theta$.

(iii) State the Rao-Blackwell theorem and see if $\bar{X}$ is UMVU for $\theta$. 

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4. Suppose that \( X_1, \ldots, X_n \) form a random sample from a uniform distribution on the interval \((0, \theta)\), and that the following hypotheses are to be tested:

\[
\begin{align*}
H_0: \quad & \theta \geq 2, \\
H_1: \quad & \theta < 2.
\end{align*}
\]

Let \( Y_n = \max (X_1, \ldots, X_n) \), and consider a test such that the critical region contains all the outcomes for which \( Y_n \leq 1.5 \).

(i) Determine the power function of the test.
(ii) Determine the size of the test.

5. Suppose that \( X_1, \ldots, X_n \) form a random sample from a normal distribution for which the mean is \( 0 \) and the variance \( \sigma^2 \) is unknown, and suppose that it is described to test the following hypotheses:

\[
\begin{align*}
H_0: \quad & \sigma^2 \leq 2, \\
H_1: \quad & \sigma^2 > 2.
\end{align*}
\]

Show that there exists a UMP test of these hypotheses at any given level of significance \( \alpha_0 \) \((0 < \alpha_0 < 1)\).

6. Suppose that \( X_1, \ldots, X_n \) form a random sample from a distribution that involves a parameter \( \theta \) whose value is unknown; and that the joint p.d.f. or the joint p.f. \( f_n (x \mid \theta) \) has a monotone likelihood ratio in the statistics \( T = r(X) \). Let \( \theta_0 \) be a specified value of \( \theta \), and suppose that the following hypotheses are to be tested:

\[
\begin{align*}
H_0: \quad & \theta \geq \theta_0, \\
H_1: \quad & \theta < \theta_0.
\end{align*}
\]

Let \( c \) be a constant such that \( \Pr(T \leq c \mid \theta = \theta_0) = \alpha_0 \). Show that the test procedure which rejects \( H_0 \) if \( T \leq c \) is a UMP test at the level of significance \( \alpha_0 \).

7. Suppose that \( X_1, \ldots, X_n \) form a random sample from a uniform distribution on the interval \((0, \theta)\), where the value of \( \theta \) is unknown; and that it is described to test the following hypotheses:

\[
\begin{align*}
H_0: \quad & \theta \leq 3, \\
H_1: \quad & \theta > 3.
\end{align*}
\]

(i) Show that for any given level of significance \( \alpha_0 \) \((0 \leq \alpha_0 < 1)\), there exists a UMP
test which specifies that $H_0$ should be rejected if $\max(X_1, \ldots, X_n) \geq c$.

(ii) Determine the value of $c$ for each possible value of $\alpha_0$.

8. Suppose that $X_1, \ldots, X_n$ form a random sample from a uniform distribution on the interval $(0, \theta)$, where the value of $\theta$ is unknown; and that it is described to test the following hypotheses:

    $H_0: \theta = 3$,
    $H_1: \theta \neq 3$.

Consider a test procedure $\delta$ such that the hypothesis $H_0$ is rejected if either $\min(X_1, \ldots, X_n) \leq c_1$ or $\max(X_1, \ldots, X_n) \geq c_2$, and let $\pi(\theta | \delta)$ denote the power function of $\delta$. Describe the values of the constants $c_1$ and $c_2$ such that $\pi(3 | \delta) = 0.05$ and $\delta$ is unbiased.