Directions: Answer as many questions as you can. One question answered completely and correctly carries more weight with the examiners than two questions that are half answered. Assertions should be accompanied by reasons; conjectures are out of place.

1. How many topologies is it possible to define on a four element set \( \{a, b, c, d\} \)? How many of these topologies are Hausdorff?

2. Let \( X \) and \( Y \) be topological spaces, and let \( f : X \to Y \) be a function. Prove that the following are equivalent:
   a) For every open subset \( U \) of \( Y \), \( f^{-1}(U) \) is open in \( X \).
   b) For every subset \( A \) of \( X \), \( f(\overline{A}) \subseteq \overline{f(A)} \).
   c) For every closed subset \( B \) of \( Y \), \( f^{-1}(B) \) is closed in \( X \).
   d) For each \( x \in X \) and each open subset \( V \) of \( Y \) with \( f(x) \in V \), there exists an open subset \( W \) of \( X \) with \( x \in W \) and \( f(W) \subseteq V \).

3. Prove that a topological space \( X \) is compact if and only if each collection \( C \) of closed subsets of \( X \) with the finite intersection property satisfies \( \bigcap_{C \in C} C \neq \emptyset \).

4. Let \( X \) be a compact Hausdorff space, and let \( A \) be a closed subset of \( X \). Prove that if \( x \in X - A \), then there are disjoint open subsets \( U, V \) of \( X \) with \( x \in U \) and \( A \subseteq V \).

5. Let \( X, Y, \) and \( Z \) be topological spaces, and let \( p : X \to Y \) be a quotient map. Let \( g : X \to Z \) be a continuous map that is constant on \( p^{-1}\{y\} \) for each \( y \in Y \). Prove that there is a continuous map \( f : Y \to Z \) such that \( g = f \circ p \).
6. Let $X$ be a topological space, and let $Y$ be a metric space, and, for each natural number $n$, let $f_n : X \to Y$ be continuous. Prove that if the sequence $(f_n)$ converges uniformly to $f : X \to Y$, then $f$ is continuous.

7. Outline a proof of Urysohn’s lemma: if $X$ is a normal space and $A, B$ are disjoint closed subsets, then there is a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

8. Let $X$ be a locally compact Hausdorff topological space.
   a) Define a compactification of $X$.
   b) Describe in detail one way to form a compactification of $X$, explaining carefully why what you describe has all the properties required of a compactification.

9. Let $X, Y,$ and $Z$ be topological spaces, with $Z$ being Hausdorff. Let $f : X \to Y$ and $g, h : Y \to Z$ be continuous functions. Show that if $f(X)$ is dense in $Y$ and $g f = h f$, then $g = h$.

10. Let $C_0 := [0, 1]$, and, for each natural number $n$, set
   
   $$C_n := C_{n-1} \setminus \bigcup_{k=0}^{\infty} \left( \frac{1 + 3k}{3^n}, \frac{2 + 3k}{3^n} \right).$$

   The intersection $C := \cap_{k=0}^{\infty} C_n$ is often called the Cantor set.
   Prove that the Cantor set is totally disconnected (that is, the only connected subsets are singletons) and compact.