Qualifying Exam in Topology January 1999

Do each problem of a separate sheet. Put your name on each sheet. If a problem requires a proof, do not assume a result that makes the problem trivial. If in doubt, ask.

1. Let \( X \) be a regular space with disjoint subsets \( A \) and \( B \) with \( A \) closed and \( B \) compact. Prove there are disjoint open sets \( U \) and \( V \), one containing \( A \) and the other containing \( B \).

2. Let \( X \) be an infinite set with the finite complement topology. For a subset \( A \) of \( X \) let \( \partial A \) be the boundary of \( A \). (boundary is called 'frontier' in some books.) Prove:
   (a) If \( A \) is finite then \( \partial A = A \)
   (b) If \( X - A \) is finite then \( \partial A = X - A \)
   (c) If neither \( A \) nor \( X - A \) is finite then \( \partial A = X \)

3. Suppose \( f : X \rightarrow Y \). One definition of continuity for \( f \) is: \( f \) is continuous if and only if whenever \( x \) is a limit point of a set \( A \) in \( X \) then \( f(x) \) is in the closure of \( f(A) \). Give another definition of continuity and prove it is equivalent to the definition above.

4. Let \( B_1, B_2, B_3, \ldots \) be a sequence of closed connected sets in a compact Hausdorff space \( X \) with \( B_n \supset B_{n+1} \) for all \( n \).
   (a) Prove \( \bigcap_{n=1}^{\infty} B_n \) is connected.
   (b) Give an example that shows that the intersection may not be connected if \( X \) is not compact.

5. Let \( \mathbb{R} \) be the real line with the usual topology, let \( Q \) be the subspace of rationals and let \( Z \) be the subspace of integers. Prove or disprove: The subspaces \( Q \) and \( Z \) are homeomorphic.

6. Prove or disprove: The product of any two connected spaces is connected.

There are more problems on the other side
7. Suppose \( X \) and \( Y \) are topological spaces and \( f : X \to Y \). Suppose also \( \mathcal{F} \) is a locally finite family of closed sets in \( X \) such that \( \bigcup \mathcal{F} = X \) and \( f|_A \) is continuous for every \( A \in \mathcal{F} \).

(a) Prove \( f \) is continuous.

(b) Give an example that shows if the family is not locally finite then the function \( f \) may not be continuous.

8. Suppose \( X \) is a first-countable space and \( E \subset X \). Show that a point \( x \) is in the closure of \( E \) if and only if there is a sequence in \( E \) that converges to \( x \).

9. Prove: In any product space, a product of closed sets is closed.

10. Prove: If \( X \) is compact, \( Y \) is Hausdorff, and \( f \) is a continuous one-to-one function from \( X \) onto \( Y \) then \( f \) is a homeomorphism.