Topology Qualifying Exam
January 2000

Do each problem of a separate sheet and put your name on each sheet.
If a problem requires a proof, do not assume a result that makes the problem trivial. If in doubt, ask. For this test, regular and normal spaces are not assumed to be $T_2$, $T_3$ and $T_4$ are assumed to be $T_2$.

1. For each positive integer $n$ let $X_n$ be the two point discrete space $\{0, 1\}$. Let $X$ be the infinite product $X_1 \times X_2 \times X_3 \times X_4 \times \ldots$ and let $0 = (0, 0, 0, 0, \ldots)$. Find a sequence in $X - \{0\}$ that converges to $0$ and prove it converges.

2. Suppose $f : X \rightarrow Y$. One definition of continuity for $f$ is: $f$ is continuous if and only if whenever $x$ is a limit point of a set $A$ in $X$ then $f(x)$ is in the closure of $f(A)$. Give another definition of continuity and prove it is equivalent to the definition above.

3. Prove: An uncountable subset of a 2nd countable space has a limit point.

4. Let $X$ be the open interval $(0, 1)$ and let $Y$ be the half open interval $[0, 1)$, both with the usual topology. Prove $X$ and $Y$ are homeomorphic.

5. Let $X$ be the closed interval $[-1, 1]$ and give $X$ the countable complement topology. (A set is open if it is empty or its complement is countable.) Answer each of the following and prove your answer.
   (a) Is $X$ Hausdorff?
   (b) Is $X$ Connected?
   (c) Let $f : X \rightarrow X$ be the absolute value function. Is $f$ continuous?

6. Let $I$ be the subspace of irrationals in the closed interval $[0, 1]$ with the usual topology. Answer each of the following and prove your answer.
   (a) Is $X$ Compact?
   (b) Is $X$ Separable?

7. Let $X$ be the real line. Define basic open sets to be all rays in $X$ of the form $[a, \infty) = \{x : x \geq a\}$. (This is a base for a topology since the for any two such rays one must be contained in the other.) Answer each of the following and prove your answer.
   (a) Is $X$ $T_0$?
   (b) Is $X$ regular?
   (c) Is $X$ normal?

8. (a) What is the fundamental group of the knot?

   (b) Use the fundamental group to prove the knot isn’t equivalent to the unknot.