CURRENCY DERIVATIVES

1 Pure Currency Contracts

Consider a situation where we have two currencies: the domestic currency (say US dollars), and the foreign currency (say pounds). The spot exchange rate at time $t$ is denoted by $X(t)$, and by definition it is quoted as

$$\frac{\text{units of the domestic currency}}{\text{unit of the foreign currency}},$$

i.e. in our example it is quoted as dollars per pound. We assume that the domestic short rate $r_d$, as well as the foreign short rate $r_f$, are deterministic constants, and we denote the corresponding riskless asset prices by $B_d$ and $B_f$ respectively. Furthermore we assume that the exchange rate is modelled by geometric Brownian motion. We can summarize this as follows.

Assumption 1
We take as given the following dynamics (under the objective probability measure $P$):

$$dX = X\alpha_X dt + X\sigma_X d\tilde{W}$$  \hspace{1cm} (1)

$$dB_d = r_d B_d dt$$  \hspace{1cm} (2)

$$dB_f = r_f B_f dt$$  \hspace{1cm} (3)

where $\alpha_X, \sigma_X$ are deterministic constants, and $\tilde{W}$ is a scalar Wiener process.

Our problem is that of pricing a currency derivative, i.e. a $T$-claim $Z$ of the form

$$Z = \Phi(X(T)),$$

European call which gives the owner the option to buy one unit of the foreign currency at the price $K$ (in the domestic currency).

First we formalize the institutional assumptions.

Assumption 2
All markets are frictionless and liquid. All holdings of the foreign currency are invested in the foreign riskless asset, i.e. they will evolve according to the dynamics

$$dB_f = r_f B_f dt.$$  

Remark
Interpreted literally this means that, for example, pounds are invested in a British bank. In reality this does not have to be the case.

Applying the standard theory of derivatives to the present situation we have the usual risk neutral valuation formula

$$\Pi(t; Z) = e^{-r_d(T-t)}E^Q_{t,x}[\Phi(X(T))],$$

and our only problem is to figure out what the martingale measure $Q$ looks like. To do this we use the result that $Q$ is characterized by the property that every domestic asset has the short rate $r_d$
as its local rate of return under \( Q \). In order to use this characterization we have to translate the possibility of investing in the foreign riskless asset into domestic terms. Since \( B_f(t) \) units of the foreign currency are worth \( B_f(t) \cdot X(t) \) in the domestic currency we immediately have the following result.

**Lemma 1**
The possibility of buying the foreign currency, and investing it at the foreign short rate of interest, is equivalent to the possibility of investing in a **domestic** asset with price process \( \tilde{B}_f \), where

\[
\tilde{B}_f(t) = B_f(t) \cdot X(t).
\]

The dynamics of \( \tilde{B}_f \) are given by

\[
d\tilde{B}_f = \tilde{B}_f(\alpha_X + r_f)dt + \tilde{B}_f \sigma_X dW.
\]

Summing up we see that our currency model is equivalent to a model of a domestic market consisting of the assets \( B_d \) and \( \tilde{B}_f \). It now follows directly from the general results that the martingale measure \( Q \) has the property that the \( Q \)-dynamics of \( \tilde{B}_f \) are given by

\[
d\tilde{B}_f = r_d \tilde{B}_f dt + \tilde{B}_f \sigma_X dW
\]

where \( W \) is a \( Q \)-Wiener process. Since by definition we have

\[
X(t) = \frac{\tilde{B}_f}{B_f(t)},
\]

we can use Itô’s formula, (3) and (4) to obtain the \( Q \)-dynamics of \( X \) as

\[
dX = X(r_d - r_f)dt + X\sigma_X dW.
\]

The basic pricing result follows immediately.

**Proposition 2 (Pricing formulas)**
The arbitrage free price \( \Pi(t; \Phi) \) for the \( T \)-claim \( Z = \Phi(X(T)) \) is given by \( \Pi(t; \Phi) = F(t, X(t)) \), where

\[
F(t, x) = e^{-r_d(T-t)}E^Q_{t,x}[\Phi(X(T))],
\]

and where the \( Q \)-dynamics of \( X \) are given by

\[
dX = X(r_d - r_f)dt + X\sigma_X dW.
\]

Alternatively \( F(t, x) \) can be obtained as the solution to the boundary value problem

\[
\begin{cases}
\frac{\partial F}{\partial t} + x(r_d - r_f)\frac{\partial F}{\partial x} + \frac{1}{2}x^2\sigma_X^2\frac{\partial^2 F}{\partial x^2} - r_d F = 0 \\
F(T, x) = \Phi(x).
\end{cases}
\]

Comparing (8) to the formulas in the last lecture, we see that a foreign currency is to be treated exactly as a stock with a continuous dividend. We may thus draw upon the results in last lecture,
which allows us to use pricing formulas for stock prices (without dividends) to price currency derivatives.

**Proposition 3 (Option pricing formula)**

Let $F_0(t, x)$ be the pricing function for the claim $Z = \Phi(X(T))$, in a world where we interpret $X$ as the price of ordinary stock without dividends. Let $F(t, x)$ be the pricing function of the same claim when $X$ is interpreted as an exchange rate. Then the following relation holds

$$F(t, x) = F_0(t, xe^{-rf(T-t)}).$$

In particular, the price of the European call, $Z = \max[X(T) - K, 0]$, on the foreign currency, is given by the modified Black-Scholes formula

$$F(t, x) = xe^{-rf(T-t)}N[d_1] - e^{-r_d(T-t)K}N[d_2],$$

where

$$d_1(t, x) = \frac{1}{\sigma_X \sqrt{T-t}} \left\{ \ln \left( \frac{x}{K} \right) + (r_d - rf + \frac{1}{2} \sigma_X^2)(T-t) \right\},$$

$$d_2(t, x) = d_1(t, x) - \sigma_X \sqrt{T-t}.$$

### 2 Domestic and Foreign Equity Markets

We will model a market which, apart from the objects of the previous section, also includes a domestic equity with (domestic) price $S_d$, and a foreign equity with (foreign) price $S_f$.

We model the equity dynamics as geometric Brownian motion, and since we now have three risky assets we use a three-dimensional Wiener process in order to obtain a complete market.

**Assumption 1**

The dynamic model of the entire economy, under the objective measure $P$, is as follows.

$$dX = X \alpha_X dt + X \sigma_X dW,$$

$$dS_d = S_d \alpha_d dt + S_d \sigma_d dW,$$

$$dS_f = S_f \alpha_f dt + S_f \sigma_f dW,$$

$$dB_d = r_d B_d dt,$$

$$dB_f = r_f B_f dt,$$

where

$$\bar{W} = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

is a three-dimensional Wiener process (as usual with independent components). Furthermore, the $(3 \times 3)$-dimensional matrix $\sigma$, given by

$$\sigma = \begin{bmatrix} \sigma_X & \sigma_{X2} & \sigma_{X3} \\ \sigma_{d1} & \sigma_{d2} & \sigma_{d3} \\ \sigma_{f1} & \sigma_{f2} & \sigma_{f3} \end{bmatrix},$$

is a...
Remark 1
The reason for the assumption about $\sigma$ is that this is the necessary and sufficient condition for completeness. It is also possible, and in many situations convenient, to model the market using three scalar correlated Wiener processes (one for each asset).

Typical $T$-contracts which we may wish to price (in terms of the domestic currency) are given by the following list.

- **A foreign equity call, struck in foreign currency**, i.e. an option to buy one unit of the foreign equity at the strike price of $K$ units of the foreign currency. The value of this claim at the date of expiration is, expressed in the foreign currency, given by

$$Z_f^T = \max[S_f(T) - K, 0]. \tag{16}$$

Expressed in terms of the domestic currency the value of the claim at $T$ is

$$Z^d = X(T) \cdot \max[S_f(T) - K, 0]. \tag{17}$$

- **A foreign equity call, struck in domestic currency**, i.e. a European option to buy one unit of the foreign equity at time $T$, by paying $K$ units of the domestic currency. Expressed in domestic terms this claim is given by

$$Z^d = \max[X(T) \cdot S_f(T) - K, 0]. \tag{18}$$

- **An Exchange option** which gives us the right to exchange one unit of the domestic equity for one unit of the foreign equity. The corresponding claim, expressed in terms of the domestic currency, is

$$Z^d = \max[X(T) \cdot S_f(T) - S_d(T), 0]. \tag{19}$$

More generally we will study pricing problems for $T$-claims of the form

$$Z = \Phi(X(T), S_d(T), S_f(T)), \tag{20}$$

where $Z$ is measured in the domestic currency. We know that the pricing function $F(t, x, s_d, s_f)$ is given by the risk neutral valuation formula

$$F(t, x, s_d, s_f) = e^{-r_d(T-t)} E_{t,x,s_d,s_f}^{Q} [\Phi(X(T))],$$

so we only have to find the correct risk adjusted measure $Q$. We transform all foreign traded assets into domestic terms. The foreign bank account has already been taken care of, and it is obvious that one unit of the foreign stock, worth $S_f(t)$ in the foreign currency, is worth $X(t) \cdot S_f(t)$ in domestic terms. We thus have the following equivalent domestic model, where the asset dynamics follow from the Ito formula.

**Proposition 4**
The original market (11)-(15) is equivalent to a market consisting of the price processes $S_d, \tilde{S}_f, \tilde{B}_f, B_d$, where

$$\tilde{B}_f(t) = X(t)B_f(t),$$

$\mu$ assumed to be invertible.
\( S_f(t) = X(t)S_f(t). \)

The \( P \)-dynamics of this equivalent model are given by

\[
\begin{align*}
    dS_d &= S_d\alpha_d dt + S_d\sigma_d d\bar{W}, \\
    d\tilde{S}_f &= \tilde{S}_f(\alpha_f + \alpha_X + \sigma_f\sigma_X^*)dt + \tilde{S}_f(\sigma_f + \sigma_X)d\bar{W}, \\
    d\tilde{B}_f &= \tilde{B}_f(\alpha_X + r_f)dt + \tilde{B}_f\sigma_X d\bar{W}, \\
    dB_d &= r_dB dt.
\end{align*}
\]

(21)

Here we have used * to denote transpose, so

\[
\sigma_f\sigma_X^* = \sum_{i=1}^{3} \sigma_{f_i}\sigma_{X_i}.
\]

Note that, because of Assumption 1, the volatility matrix above is invertible, so the market is complete.

Since \( S_d, \tilde{S}_f, \tilde{B}_f \) can be interpreted as prices of domestically traded assets, we can easily obtain the relevant \( Q \)-dynamics.

**Proposition 5**

The \( Q \)-dynamics are as follows.

\[
\begin{align*}
    dS_d &= S_d\alpha_d dt + S_d\sigma_d dW, \\
    d\tilde{S}_f &= \tilde{S}_f(\alpha_f + \alpha_X + \sigma_f\sigma_X^*)dt + \tilde{S}_f(\sigma_f + \sigma_X)dW, \\
    d\tilde{B}_f &= \tilde{B}_f(\alpha_X + r_f)dt + \tilde{B}_f\sigma_X dW, \\
    dB_d &= r_dB dt.
\end{align*}
\]

(25)

We can now immediately obtain the risk neutral valuation formula, and this can in fact be done in two ways. We can either use \( (X, S_d, \tilde{S}_f) \) as state variables, or use the equivalent (there is one-to-one mapping) set \( (X, S_d, \tilde{S}_f) \). Which set to use is a matter of convenience, depending on the particular claim under study, but in both cases the arbitrage free price is given by the discounted expected value of the claim under the \( Q \)-dynamics.

**Proposition 6 (Pricing formulas)**

For a claim of the form

\[ Z = \Phi(X(T), S_d(T), \tilde{S}_f(T)) \]

the corresponding pricing function \( F(t, x, s_d, \tilde{s}_f) \) is given by

\[
F(t, x, s_d, \tilde{s}_f) = e^{-r_d(T-t)}E^Q_t[\Phi(X(T)), S_d(T), \tilde{S}_f(T)],
\]

(30)

where the \( Q \)-dynamics are given by Proposition 5.
The pricing PDE is
\[ \frac{\partial F}{\partial t} + x(r_d - r_f) \frac{\partial F}{\partial x} + s_d r_d \frac{\partial F}{\partial s_d} + s_f r_f \frac{\partial F}{\partial s_f} \]
\[ + \frac{1}{2} \left\{ x^2 \| \sigma_X \|^2 \frac{\partial^2 F}{\partial x^2} + s_d^2 \| \sigma_d \|^2 \frac{\partial^2 F}{\partial s_d^2} + s_f^2 (\| \sigma_f \|^2 + \| \sigma_X \|^2 + 2 \sigma_f \sigma_X^* \sigma_d^* \sigma_d^* \frac{\partial^2 F}{\partial s_f^2} \right\} \]
\[ + s_d x \sigma_d^* \frac{\partial^2 F}{\partial s_d \partial x} + s_f x (\sigma_f^* \sigma_X^* + \| \sigma_X \|^2) \frac{\partial^2 F}{\partial s_f \partial x} \]
\[ + s_d s_f (\sigma_d^* \sigma_f^* + \sigma_d^* \sigma_f^* \sigma_X^* \sigma_d^* \sigma_d^* \frac{\partial^2 F}{\partial s_d \partial s_f} - r_d F = 0 \]
\[ F(T, x, s_d, s_f) = \Phi(x, s_d, s_f) \]

**Proposition 7**
For a claim of the form
\[ Z = \Phi(X(T), S_d(T), S_f(T)) \]
the corresponding pricing function \( F(t, x, s_d, s_f) \) is given by
\[ F(t, x, s_d, s_f) = e^{-r_d(T-t)} E^Q_{t,x,s_d,s_f} [\Phi(X(T)), S_d(T), S_f(T)], \] (31)
where the \( Q \)-dynamics are given by Proposition 5.

The pricing PDE is
\[ \frac{\partial F}{\partial t} + x(r_d - r_f) \frac{\partial F}{\partial x} + s_d r_d \frac{\partial F}{\partial s_d} + s_f (r_f - \sigma_f \sigma_X^*) \frac{\partial F}{\partial s_f} \]
\[ + \frac{1}{2} \left\{ x^2 \| \sigma_X \|^2 \frac{\partial^2 F}{\partial x^2} + s_d^2 \| \sigma_d \|^2 \frac{\partial^2 F}{\partial s_d^2} + s_f^2 (\| \sigma_f \|^2 + 2 \sigma_f \sigma_X^* \sigma_d^* \sigma_d^* \frac{\partial^2 F}{\partial s_f^2} \right\} \]
\[ + s_d x \sigma_d^* \frac{\partial^2 F}{\partial s_d \partial x} + s_f x (\sigma_f^* \sigma_X^* + \| \sigma_X \|^2) \frac{\partial^2 F}{\partial s_f \partial x} \]
\[ + s_d s_f (\sigma_d^* \sigma_f^* + \sigma_d^* \sigma_f^* \sigma_X^* \sigma_d^* \sigma_d^* \frac{\partial^2 F}{\partial s_d \partial s_f} - r_d F = 0 \]
\[ F(T, x, s_d, s_f) = \Phi(x, s_d, s_f) \]

**Remark 2.3**
In many applications the claim under study is of the restricted form
\[ Z = \Phi(X(T), S_f(T)). \]
In this case all partial derivatives w.r.t. \( s_d \) vanish from the PDEs above. A similar reduction will of course also take place for a claim of the form
\[ Z = \Phi(X(T), S_d(T)). \]

**Remark 2.4**
In practical applications it may be more convenient to model the market, and easier to read the
formulas above, if we model the market using correlated Wiener processes. We can formulate our basic model (under \( Q \)) as

\[
\begin{align*}
    dX &= X\alpha_X dt + X\delta_X d\bar{V}_X, \\
    dS_d &= S_d\alpha_d dt + S_d\delta_d d\bar{V}_d, \\
    dS_f &= S_f\alpha_f dt + S_f\delta_f d\bar{V}_f, \\
    dB_d &= r_d B_d dt, \\
    dB_f &= r_f B_f dt.
\end{align*}
\]

where the three processes \( \bar{V}_X, \bar{V}_d, \bar{V}_f \) are one-dimensional corelated Wiener processes. We assume that \( \delta_X, \delta_d, \delta_f \) are positive. The instantaneous correlation between \( \bar{V}_X \) and \( \bar{V}_f \) is denoted by \( \rho_{Xf} \) and correspondingly for the other pairs. We then have the following set of translation rules between the two formalisms

\[
\begin{align*}
    ||\sigma_i|| &= \delta_i, & i &= X, d, f, \\
    \sigma_i\sigma_j^* &= \delta_i\delta_j\rho_{ij}, & i, j &= X, d, f, \\
    ||\sigma_i + \sigma_j|| &= \sqrt{\delta_i^2 + \delta_j^2 + 2\delta_i\delta_j\rho_{ij}}, & i, j &= X, d, f.
\end{align*}
\]

3 Domestic and Foreign Market Prices of Risk

This section constitutes a small digression in the sense that we will not derive any new pricing formulas. Instead we will take a closer look at the various market prices of risk. As will be shown below, we have to distinguish between the domestic and the foreign market price of risk, and we will clarify the connection between these two objects. As a by-product we will obtain a somewhat deeper understanding of the concept of risk neutrality.

Let us therefore again consider the international model \( X, S_d, S_f, B_d, B_f \), with dynamics under the objective measure \( P \) given by (11)-(15). As before we transform the international model into the domestically traded assets \( S_d, \tilde{S}_f, B_d, \tilde{B}_f \) with \( P \)-dynamics given by (21)-(24).

In the previous section we infer the existence of a martingale measure \( Q \), under which all domestically traded assets command the domestic short rate \( r_d \) as the local rate of return. Our first observation is that, from a logical point of view, we could just as well have chosen to transform (11)-(15) into equivalent assets traded on the foreign market. Thus we should really denote our “old” martingale measure \( Q \) by \( Q_d \) in order to emphasize its dependence on the domestic point of view. If we instead take a foreign investor’s point of view we will end up with a “foreign martingale measure”, which we will denote by \( Q_f \), and an obvious project is to investigate the relationship between these martingale measures. A natural guess is perhaps that \( Q_d = Q_f \), but as we shall see this is generically not the case. Since there is of risk, we will carry out the project above in terms of market prices of risk.

We start by taking the domestic point of view, and applying the previous result to the domestic price processes (21)-(24), we infer the existence of the domestic market price of risk process

\[
\lambda_d(t) = \begin{bmatrix} \lambda_{d1}(t) \\ \lambda_{d2}(t) \\ \lambda_{d3}(t) \end{bmatrix}
\]
with the property that if $\Pi$ is the price process of any domestically traded asset in the model, with price dynamics under $P$ of the form

$$d\Pi(t) = \Pi(t)\alpha(t)dt + \Pi(t)\sigma(t)d\bar{W}(t),$$

then, for all $t$ and $P$-a.s., we have

$$\alpha(t) - r_d = \alpha(t)\lambda_d(t).$$

Applying this to (21)-(23) we get the following set of equations:

$$\alpha_d - r_d = \sigma_d \cdot \lambda_d, \quad (32)$$
$$\alpha_f + \alpha_X + \sigma_f \sigma_X^* - r_d = (\sigma_f + \sigma_X)\lambda_d, \quad (33)$$
$$\alpha_X + r_f - r_d = \sigma_X \cdot \lambda_d. \quad (34)$$

In passing we note that, since the coefficient matrix

$$\sigma = \begin{bmatrix} \sigma_d \\ \sigma_f + \sigma_X \\ \sigma_X \end{bmatrix}$$

is invertible by Assumption 2.1, $\lambda_d$ is uniquely determined (and in fact constant). This uniqueness is of course equivalent to the completeness of the model.

We now go on to take the perspective of a foreign investor, and the first thing to notice is that the model (11)-(15) of the international market does not treat the foreign and the domestic points of view symmetrically. This is due to the fact that the exchange rate $X$ by definition is quoted as

\begin{align*}
\text{units of the domestic currency} \\
\text{unit of the foreign currency}
\end{align*}

From the foreign point of view the exchange rate $X$ should thus be replaced by the exchange rate $Y(t) = \frac{1}{X(t)}$ which is then quoted as

\begin{align*}
\text{unit of the foreign currency} \\
\text{units of the domestic currency}
\end{align*}

and the dynamics for $X,S_d,S_f,B_d,B_f$ should be replaced by the dynamics for $Y,S_d,S_f,B_d,B_f$. In order to do this we only have to compute the dynamics of $Y$, given those of $X$, and an easy application of Ito’s formula gives us

$$dY = Y\alpha_Y dt + Y\sigma_Y d\bar{W}, \quad (35)$$

where

$$\alpha_Y = -\alpha_X + ||\sigma_X||^2, \quad (36)$$
$$\alpha_Y = -\alpha_X. \quad (37)$$
Following the arguments the domestic analysis, we now transform the processes $Y, S_d, S_f, B_d, B_f$ into a set of asset prices on the foreign market, namely $S_f, \tilde{S}_d, \tilde{B}_d$, where

\[
\tilde{S}_d = Y \cdot S_d, \\
\tilde{B}_d = Y \cdot B_d.
\]

If we want to obtain the $P$-dynamics of $S_f, \tilde{S}_d, \tilde{B}_d$ we noe only have to use (21)-(24), substituting $Y$ for $X$ and $d$ for $f$. Since we are not interested in these dynamics per se, we will, however, not carry out these computations. The object that we are primarily looking for is the foreign market price of risk $\lambda_f$, and we can easily obtain that by writing down the foreign version of the system and substituting $d$ for $f$ and $Y$ for $X$ directly in (32)-(34). We get

\[
\alpha_f - r_f = \sigma_f \cdot \lambda_f, \\
\alpha_d + \alpha_Y + \sigma_d \sigma_Y^* - r_f = (\sigma_d + \sigma_Y) \lambda_f, \\
\alpha_Y + r_d - r_f = \sigma_Y \cdot \lambda_f.
\]

and inserting (36)-(37), we finally obtain

\[
\alpha_f - r_f = \sigma_f \cdot \lambda_f, \quad (38) \\
\alpha_d - \alpha_X + ||\sigma_X||^2 - \sigma_d \sigma_X^* - r_f = (\sigma_d - \sigma_X) \lambda_f, \quad (39) \\
-\alpha_X + ||\sigma_X||^2 + r_d - r_f = -\sigma_X \cdot \lambda_f. \quad (40)
\]

After some simple algebraic manipulations, the two systems (32)-(34) and (38)-(40) can be written as

\[
\alpha_X + r_f - r_d = \sigma_X \cdot \lambda_d, \\
\alpha_d - r_d = \sigma_d \cdot \lambda_d, \\
\alpha_f + \sigma_f \sigma_X^* - r_f = \sigma_f \cdot \lambda_d, \\
\alpha_X - ||\sigma_X||^2 + r_f - r_d = \sigma_X \cdot \lambda_f, \\
\alpha_d - \sigma_d \sigma_X^* - r_d = \sigma_d \cdot \lambda_f, \\
\alpha_f - r_f = \sigma_f \cdot \lambda_f.
\]

These equations can be written as

\[
\delta = \sigma \lambda_d, \\
\varphi = \sigma \lambda_f,
\]

where

\[
\delta = \begin{bmatrix}
\alpha_X + r_f - r_d \\
\alpha_d - r_d \\
\alpha_f + \sigma_f \sigma_X^* - r_f
\end{bmatrix}, \quad \varphi = \begin{bmatrix}
\alpha_X - ||\alpha_X||^2 + r_f - r_d \\
\alpha_d - \sigma_d \sigma_X^* - r_d \\
\alpha_f - r_f
\end{bmatrix}
\]

So, since $\sigma$ is invertible,

\[
\lambda_d = \sigma^{-1} \delta,
\]

\[9\]
Thus we have
\[ \lambda_f = \sigma^{-1}\varphi. \]

Hence we have
\[ \lambda_d - \lambda_f = \sigma^{-1}(\delta - \varphi), \]
and since
\[ \delta - \varphi = \begin{bmatrix} \sigma_X\sigma_X^* \\ \sigma_d\sigma_X^* \\ \sigma_f\sigma_X^* \end{bmatrix} = \sigma\sigma_X^*, \]
we obtain
\[ \lambda_d - \lambda_f = \sigma^{-1}(\delta - \varphi) = \sigma^{-1}\sigma\sigma_X^* = \sigma\sigma_X^*. \]
We have thus proved the following central result.

**Proposition 10**
The foreign market price of risk is uniquely determined by the domestic market price of risk, and by the exchange rate volatility vector \( \sigma_X \), through the formula
\[ \lambda_f = \lambda_d - \sigma\sigma_X^*. \tag{41} \]

**Remark 3.1**
For the benefit of the probabilist we note that this result implies that the transition from \( Q_d \) to \( Q_f \) is effected via a Girsanov transformation, for which the likelihood process \( L \) has the dynamics
\[ dL = L\sigma_XdW, \]
\[ L(0) = 1. \]

Proposition 10 has immediately consequences for the existence of risk neutral markets. If we focus on the domestic market can we say that the market is (on the aggregate) risk neutral if the following valuation formula holds, where \( \Pi_d \) is the price process for any domestically traded asset.
\[ \Pi_d(t) = e^{-r_d(T-t)}E^P[\Pi_d(T)|\mathcal{F}_t]. \tag{42} \]
In other words, the domestic market is risk neutral if and only if \( P = Q_d \). In many scientific papers an assumption is made that the domestic market is in fact risk neutral, and this is of course a behavioral assumption, typically made in order to facilitate computations. In an international setting it then seems natural to assume that both the domestic market and the foreign market are risk neutral, i.e. that, in addition to (42), the following formula also holds, where \( \Pi_f \) is the foreign price of any asset traded on the foreign market
\[ \Pi_f(t) = e^{-r_f(T-t)}E^P[\Pi_f(T)|\mathcal{F}_t]. \tag{43} \]
This seems innocent enough, but taken together these assumptions imply that
\[ P = Q_d = Q_f. \tag{44} \]
Proposition 10 now tells us that (44) can never hold, unless \( \sigma_X = 0 \), i.e. if and only if the exchange rate is deterministic.
At first glance this seems highly counter-intuitive, since the assumption about risk neutrality often is interpreted as an assumption about the (aggregate) attitude towards risk as such. However, from (42), which is an equation for objects measured in the domestic currency, it should be clear that risk neutrality is a property which holds only relative to a specified numeraire. To put it as a slogan, you may vary well be risk neutral w.r.t. pounds sterling, and still be risk averse w.r.t. US dollars.

There is nothing very deep going on here: it is basically just the Jensen inequality. To see this more clearly let us consider the following simplified situation. We assume that \( r_d = r_f = 0 \), and we assume that the domestic market is risk neutral. This means in particular that the exchange rate itself has the following risk neutral valuation formula

\[
X(0) = E[X(T)]. \tag{45}
\]

Looking at the exchange rate from the foreign perspective we see that if the foreign market also is risk neutral, then it must hold that

\[
Y(0) = E[Y(T)], \tag{46}
\]

with \( Y = 1/X \). The Jensen inequality together with (45) gives us, however,

\[
Y(0) = \frac{1}{X(0)} = \frac{1}{E[X(T)]} \leq E[\frac{1}{X(T)}] = E[Y(T)]. \tag{47}
\]

Thus (45) and (46) can never hold simultaneously with a stochastic exchange rate.