Lecture 6

Infinitesimal generator of a diffusion

Let’s look at
\[ dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) \]

Definition

The infinitesimal generator of \( X \) is the 2\(^{nd} \) order differential operator \( A = A_t \) defined by:

\[
(A_t F)(x) = (AF)(t, x) = \mu(t, x) \cdot \nabla F(x) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\sigma \sigma^*)_{i,j}(t, x) \frac{\partial^2 F}{\partial x_i \partial x_j}(x)
\]

Question: What does this \( A \) have to do with \( X \)?

Connection to PDEs

We showed last time that

\[
(1) \quad F(t, X(t)) = F(0, X(0)) + \int_0^t \frac{\partial F}{\partial s}(s, X(s))ds + \int_0^t (AF)(s, X(s))ds + \text{Mart}(t)
\]

Now assume that \( F \) solves the following problem:

\[
(2) \quad \begin{cases} 
F(T, x) = \phi(x) & \forall x \in \mathbb{R}^n \\ 
\frac{\partial F}{\partial t}(t, x) + A F(t, x) = 0 & \forall t \leq T, \forall x \in \mathbb{R}^n 
\end{cases}
\]

Observe that this equation does not have anything random. It is the heat equation backwards in time, start at time \( T \), go back in time.

So we proved the following theorem:

Theorem:

If a solution \( F \) to equation(2)

\[
\begin{cases} 
F(T, x) = \phi(x) & \forall x \in \mathbb{R}^n \\ 
\frac{\partial F}{\partial t}(t, x) + A F(t, x) = 0 & \forall t \leq T, \forall x \in \mathbb{R}^n 
\end{cases}
\]

exists and is sufficiently smooth, then

\[ F(t, x) = E_{t,x}[\phi(X(T))] \]

Where \( X \) is the diffusion with generator \( A \), started at \( X(t) = x \), i.e. \( X \) satisfies (1) on the interval \([t, T]\).
Remark:
This also constitutes a proof for uniqueness of solution of such problems.

Feynman-Kac formula

For solving the linear multiplicative parabolic PDE:

Assume $F$ is the solution of the following boundary problem:

$$
\begin{align*}
F(T, x) &= \phi(x) \quad \forall x \in \mathbb{R} \\
\frac{\partial F}{\partial t}(t, x) + AF(t, x) + F(t, x)V(t, x) &= 0 \quad \forall t \leq T, \forall x \in \mathbb{R}
\end{align*}
$$

Then: $F(t, x) = E_{t,x}[\phi(X(T))e^{\int_t^T V(s,X(s))ds}]$

Ex 1:

Find the solution for:

$$
\begin{align*}
F(T, x) &= \phi(x) \\
\frac{\partial F}{\partial t}(t, x) + AF(t, x) - rF(t, x) &= 0
\end{align*}
$$

Ex 2: 5.9