Financial market instruments can be divided into two types. There are the underlying stocks – shares, bonds, commodities, foreign currencies; and their derivatives, claims that promise some payment or delivery in the future contingent on an underlying stock’s behavior. Derivatives can reduce risk – by enabling a player to fix a price for a future transaction now – or they can magnify it. A costless contract agreeing to pay off the difference between a stock and some agreed future price lets both sides ride the risk inherent in owning a stock, without needing the capital to buy it outright.

**Definition**

A *forward contract* is an agreement to buy (or sell) an asset on a specified future date, $T$, for a specified price, $K$. The buyer is said to hold the *long* position, the seller the *short* position.

Forwards are not generally traded on exchanges. It costs nothing to enter into a forward contract. The ‘pricing problem’ for a forward is to determine what value of $K$ should be written into the contract. A *futures contract* is the same as a forward except that future *are* normally traded on exchanges and the exchange specifies certain standard features of the contract and a particular form of settlement.

Forwards provide the simplest examples of derivative securities and the mathematics of the corresponding pricing problem will also be simple. A much richer theory surrounds the pricing of *options*. An option gives the holder the *right*, but not the *obligation*, to do something. Options come in many different guises. Black and Scholes gained fame for pricing a European call option.

**Definition**

A *European call option* gives the holder the right, but not the obligation, to buy an asset at a specified time, $T$, for a specified price, $K$. A *European put option* gives the holder the right to sell an asset for a specified price, $K$, at time $T$.

In general *call* refers to buying and *put* to selling. The term *European* is reserved for options whose value to the holder at the time, $T$, when the contract expires depends on the state of the market only at time $T$. There are other options, for example American options or Asian options, whose payoff is contingent on the behavior of the underlying over the whole time interval $[0, T]$.

**Definition**

The time, $T$, at which the derivative contract expires is called the *exercise date* or the *maturity*. The price $K$ is called the *strike price*.

For a company that needs oil to function (an airline, for example), one can think of the option as insurance against increasing oil prices. The pricing problem is now to determine, for given $T$ and $K$, how much the company should be willing to pay for such insurance.

For this example there is an extra complication: it costs money to store oil. To simplify our task we are first going to price derivatives based on assets that can be held without additional cost,
typically company shares. Equally we suppose that there is no additional benefit to holding the shares, that is no dividends are paid.

As a first step, we need to know what the contract will be worth at the expiry date. If at the time when the option expires the actual price of the underlying stock is $S_T$ and $S_T > K$ then the option will be exercised. The option is then said to be in the money: an asset worth $S_T$ can be purchased for just $K$. The value to the company of the option is then $(S_T - K)$. If, on the other hand, $S_T - K < 0$, then it will be cheaper to buy the underlying stock on the open market and so the option will not be exercised. (It is this freedom not to exercise that distinguishes options from futures.) The option is then worthless and is said to be out of the money. (If $S_T = K$ the option is said to be at the money.) The payoff of the option at time $T$ is thus

$$(S_T - K)_+ \triangleq \max\{(S_T - K, 0)\}$$

We have presented the European call option as a means of reducing risk. Of course it can also be used by a speculator as a bet on an increase in the stock price. In fact by holding packages, that is combinations of the ‘vanilla’ options that we have described so far, we can take rather complicated bets. We present just one example.

**Example (A straddle)**

Suppose that a speculator is expecting a large move in a stock price, but does not know in which direction that move will be. Then a possible combination is a straddle. This involves holding a European call and a European put with the same strike price and maturity.

**Explanation:**

The payoff of this straddle is $(S_T - K)_+$ (from the call) plus $(K - S_T)_+$ (from the put), that is, $|S_T - K|$. Although the payoff of this combination is always positive, if, at the expiry time, the stock price is too close to the strike price then the payoff will not be sufficient to offset the cost of purchasing the options and the investor makes a loss. On the other hand, large movements in price can lead to substantial profits.

**Pricing a forward**

In order to solve our pricing problems, we are going to have to make some assumptions about the way in which markets operate. To formulate these we begin by discussing forward contracts in more detail.

At the time when the contract is written, we don’t know $S_T$, we can only guess at it, or, more formally, assign a probability distribution to it. A widely used model (which underlies the Black-Scholes analysis) is that stock prices are lognormally distributed. That is, there are constants $v$ and $\sigma$ such that the logarithm of $S_T/S_0$ (the stock price at time $T$ divided by that at time zero, usually called the return) is normally distributed with mean $v$ and variance $\sigma^2$. In symbols:

$$P\left[\frac{S_T}{S_0} \in [a, b]\right] = P[\log(\frac{S_T}{S_0}) \in [\log a, \log b]] = \int_{\log a}^{\log b} \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(x - v)^2}{2\sigma^2}\right) dx$$

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Notice that stock prices, and therefore \( a \) and \( b \), should be positive, so that the integral on the right hand side is well defined.

Our first guess might be that \( \mathbb{E}[S_T] \) should represent a fair price to write into our contract. However, it would be a rare coincidence for this to be the market price. In fact we’ll show that the cost of borrowing is the key to our pricing problem.

We need a model for the time value of money: a dollar now is worth more than a dollar promised at some later time. We assume a market for these future promises (the bond market) in which prices are derivable from some interest rate. Specifically:

**Time value of money**

We assume that for any time \( T \) less than some horizon \( \tau \) the value now of a dollar promised at \( T \) is \( e^{-rT} \) for some constant \( r > 0 \). The rate \( r \) is then the continuously compounded interest rate for this period.

Such a market, derived from say US Government bonds, carries no risk of default – the promise of a future dollar will always be honored. To emphasis this we will often refer to \( r \) as the risk−free interest rate. In this model, by buying or selling cash bonds, investor can borrow money for the same risk-free rate of interest as they can lend money.

Interest rate markets are not this simple in practice, but that is an issue that we shall defer.

We now show that it is the risk−free interest rate, or equivalently the price of a cash bond, and not our lognormal model that forces the choice of the strike price, \( K \), upon us in our forward contract.

Interest rates will be different for different currencies and so, for definiteness, suppose that we are operating in the dollar market, where the (risk-free) interest rate is \( r \).

- Suppose first that \( K > S_0 e^{rT} \). The seller, obliged to deliver a unit of stock for \( \$K \) at time \( T \), adopts the following strategy: she borrows \( \$S_0 \) at time zero (i.e. sells bonds to the value \( \$S_0 \)) and buys one unit of stock. At time \( T \), she must repay \( \$S_0 e^{rT} \), but she has the stock to sell for \( \$K \), leaving her a certain profit of \( \$(K − S_0 e^{rT}) \).

- If \( K < S_0 e^{rT} \), then the buyer reverses the strategy. She sells a unit of stock at time zero for \( \$S_0 \) and buys cash bonds. At time \( T \), the bonds deliver \( \$S_0 e^{rT} \) of which she uses \( \$K \) to buy back a unit of stock leaving her with a certain profit of \( \$(S_0 e^{rT} − K) \).

Unless \( K = S_0 e^{rT} \), one party is guaranteed to make a profit.

**Definition**

An opportunity to lock into a risk-free profit is called an **arbitrage** opportunity.

The starting point in establishing a model in modern finance theory is to specify that there is no arbitrage. (in fact there are people who make their living entirely from exploiting arbitrage opportunities, but such opportunities do not exist for a significant length of time before market
prices move to eliminate them.) We have proved the following lemma.

**Lemma**

In the absence of arbitrage, the strike price in a forward contract with expiry date $T$ on a stock whose value at time zero is $S_0$ is $K = S_0 e^{rT}$, where $r$ is the risk-free rate of interest.

The price $S_0 e^{rT}$ is sometimes called the *arbitrage price*. It is also known as the *forward price* of the stock.

**Remark**

In our proof of this Lemma, the buyer sold stock that she may not own. This is known as *short selling*. This can, and does, happen: investors can ‘borrow’ stock as well as money.

Of course forwards are a very special sort of derivative. The argument above won’t tell us how to value an option, but the strategy of seeking a price that does not provide either party with a risk-free profit will be fundamental in what follows.

Let us recap what we have done. In order to price the forward, we constructed a portfolio, comprising one unit of underlying stock and $-S_0$ cash bonds, whose value at the maturity time $T$ is exactly that of the forward contract itself. Such a portfolio is said to be a *perfect hedge* or *replicating portfolio*. This idea is the central paradigm of modern mathematical finance and will recur again and again in what follows. Ironically we shall use expectation repeatedly, but as a tool in the construction of a perfect hedge.

**Pricing an option: a discrete model**

Model specification (simple period model)

We fix the following notations:

1) There are 2 times (dates) $t = 0$ and $t = 1$. We could trade (or consume) at these 2 dates.

2) There are $k$ possible states of the world, the value of which is unknown at time $t = 0$, but it is known at time $t = 1$. We write this in a sample space.

$$\Omega = \{\omega_1, \omega_2, ..., \omega_k\}$$

3) A probability measure $P$ on $\Omega$, with $P(\omega) > 0$ for all $\omega \in \Omega$ and $\sum_{i=1}^{k} P(\omega_i) = 1$ is given.

4) A price process $S = \{S_t, t = 0, 1\}$ where $S_t = (S_1(t), S_2(t), ..., S_N(t))$, $N < \infty$ and $S_n(t)$ is the time $t$ price of security $n$.

**Remarks:**

a) These risky securities are most of the time stocks.

b) At time $t = 0$ the prices are known to the investor and they are positive scalars.
c) At time \( t = 1 \) the prices are non-negative random variables whose values become known to the investor only at time \( t = 1 \).

5) There is a bank account process \( B = \{B_t, t = 0, 1\} \) where \( B_0 = 1 \) and \( B_1 \) is a random variable. For the bank account \( B_1(\omega) > 0 \) for all \( \omega \)'s. This is different from the risky securities where \( S_1(\omega) \) could be 0. Usually, in fact, \( B_1 \geq 1 \), but it is not necessary. We could look at \( B_1 \) as being the time \( t = 1 \) value of a bank account of $1 at time \( t = 0 \). Then the interest rate is \( r = B_1 - 1 \) and there are models where we allow \( r \leq 0 \). Nevertheless \( B_1 > 0 \).

**Definition**

A trading strategy, denoted by \( H = (H_0, H_1, ..., H_N) \) describes an investor’s portfolio as carried forward from time \( t = 0 \) to time \( t = 1 \).

\( H_0 \) = # of dollars invested in the bank.
\( H_n \) = # of units of security \( n \) held between \( t = 0 \) and \( t = 1 \).

**Remarks:**

a) \( H_n \) could be negative, or positive.

b) If \( H_n \) is negative that means that we borrow (if \( n = 0 \)) or we are selling short, or have a short position in asset \( n \). If \( H_n \) is positive then we say we have a long position in asset \( n \).

**Definition**

The value process \( V = \{V_t, t = 0, 1\} \) is given by the total value of the portfolio at each point in time, i.e.

\[
V_t = H_0B_t + \sum_{n=1}^{N} H_nS_n(t), \ (t = 0, 1)
\]

or \( V_0 = H_0B_0 + \sum_{n=1}^{N} H_nS_n(0) \)

\( V_1 = H_0B_1 + \sum_{n=1}^{N} H_nS_n(1) \)

\( V_1 - V_0 = H_0(B_1 - B_0) + \sum_{n=1}^{N} H_n(S_n(1) - S_n(0)) \)

\( V_1 - V_0 = H_0r + \sum_{n=1}^{N} H_n\Delta S_n \)

This quantity is called the gain process \( G = V_1 - V_0 \) and it is a random variable. \( G \) is the total profit or loss generated by the portfolio between times \( t = 0 \) and \( 1 \)

(We do not include addition of funds or consumption!)

**Definition**
The discounted price process $S^* = \{S^*_t, t = 0, 1\}$, $S^*_t = (S^*_1(t), ..., S^*_N(t))$ is defined by

$$S^*_n(t) = \frac{S_n(t)}{B_t} \quad n = 1, ..., N, \quad t = 0, 1$$

We normalize the prices such that the bank becomes constant (is the numeraire)

So

$$V^*_t = H_0 + \sum_{n=1}^{N} H_n S^*_n(t)$$

and the discounted gains process:

$$G^* = \sum_{n=1}^{N} \Delta S^*_n$$

Observe that $V^*_t B_t = V_t \implies V^*_t = \frac{V_t}{B_t}$ and $G^* = V^*_1 - V^*_0$.

**Arbitrage**

**Definition**

An opportunity of making a profit on a transaction without being expose to the risk of incurring a loss is called an arbitrage opportunity. Formally, an arbitrage opportunity is some strategy $H$ such that

(a) $V_0 = 0$

(b) $V_1 \geq 0$

(c) $EV_1 > 0$

**Remark**

1) This is a riskless way of making money. You start with 0 money and without a chance of going into debt (debt means $V_1 < 0$). There is a chance of ending up with a positive amount of money ($EV_1 > 0$).

2) An economical model with arbitrage opportunities will not be in equilibrium, therefore we are interested in models (that are interesting from economic standpoint) that don’t have arbitrage opportunities. The question that we will explore is: what are the condition that have to be satisfied in order for the model to be free of arbitrage. There is no easy way to check directly whether a model has any arbitrage opportunities, but there are necessary and sufficient conditions for the model to be free of arbitrage.

3) $H$ is an arbitrage opportunity if and only if:

(a) $G^* \geq 0$

(b) $EG^* > 0$

(c) $V_0^* = 0$

To see this, consider $H$ – an arbitrage opportunity. we saw

$$G^* = V^*_1 - V^*_0 = V^*_1 - 0 \geq 0$$
\[ EG^* = EV_1^* - EV_0^* = EV_1^* - 0 = EV_1^* > 0 \]

So

\[ G^* \geq 0, \quad EG^* > 0, \quad V_0^* = 0. \]

Conversely, suppose there is a strategy \( \hat{H} \) such that \( G^* \geq 0 \) and \( EG^* > 0 \). Consider the strategy

\[ H = (H_0, \hat{H}_1, \ldots, \hat{H}_N) \quad \text{where} \quad H_0 = -\sum_{n=1}^{N} \hat{H}_n S_n^*(0). \]

Then \( V_0^* = 0 \) and \( V_1^* = V_0^* + G^* = G^* \geq 0 \). Also, \( EV_1^* = EG^* > 0 \). So \( H \) is an arbitrage opportunity.