Lecture 10

Time depending Ito formula:

\[ d[f(t, X(t))] = \frac{\partial f}{\partial X}(t, X(t)) dX(t) + \frac{\partial f}{\partial t}(t, X(t)) dt + \frac{1}{2} \frac{\partial^2 f}{\partial X^2}(t, X(t)) d < X > (t) \]

Multidimensional Ito formula

For \( i = 1, 2, ..., n, \)

\[ dX_i(t) = \mu_i(t) dt + \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t) \]

where \( \mu_i, \sigma_{ij} \) are adapted processes and \( \{W_j, j = 1, ..., d\} \) are iid BMs.

So we have \( n \) semimartingales \( X_i \) and each of them might depend on “d” BMs (\( W_j \))

Remark:

The semimartingales \( X_i \) are not independent, because they all depend on the same noise terms \( W_j \).

How do we write this in matrix notation?

\[
\begin{align*}
\frac{dX_1(t)}{dt} &= \mu_1(t) dt + \sigma_{11}(t) dW_1(t) + \sigma_{12}(t) dW_2(t) + \sigma_{13}(t) dW_3(t) + \cdots + \sigma_{1d}(t) dW_d(t) \\
\frac{dX_2(t)}{dt} &= \mu_2(t) dt + \sigma_{21}(t) dW_1(t) + \sigma_{22}(t) dW_2(t) + \sigma_{23}(t) dW_3(t) + \cdots + \sigma_{2d}(t) dW_d(t) \\
\frac{dX_3(t)}{dt} &= \mu_3(t) dt + \sigma_{31}(t) dW_1(t) + \sigma_{32}(t) dW_2(t) + \sigma_{33}(t) dW_3(t) + \cdots + \sigma_{3d}(t) dW_d(t) \\
& \quad \vdots \\
\frac{dX_n(t)}{dt} &= \mu_n(t) dt + \sigma_{n1}(t) dW_1(t) + \sigma_{n2}(t) dW_2(t) + \sigma_{n3}(t) dW_3(t) + \cdots + \sigma_{nd}(t) dW_d(t)
\end{align*}
\]

Denote by \( dX(t) \) the vector

\[
\begin{bmatrix}
\frac{dX_1(t)}{dt} \\
\frac{dX_2(t)}{dt} \\
\frac{dX_3(t)}{dt} \\
\vdots \\
\frac{dX_n(t)}{dt}
\end{bmatrix}
\]

\( -n \times 1 \) matrix; \( \mu(t) =
\begin{bmatrix}
\mu_1(t) \\
\mu_2(t) \\
\mu_3(t) \\
\vdots \\
\mu_n(t)
\end{bmatrix}
\]

\( -n \times 1 \) matrix

\( dW(t) =
\begin{bmatrix}
\frac{dW_1(t)}{dt} \\
\frac{dW_2(t)}{dt} \\
\frac{dW_3(t)}{dt} \\
\vdots \\
\frac{dW_d(t)}{dt}
\end{bmatrix}
\]

\( -d \times 1 \) matrix , and \( \sigma(t) =
\begin{bmatrix}
\sigma_{11}(t) & \sigma_{12}(t) & \cdots & \sigma_{1d}(t) \\
\sigma_{21}(t) & \sigma_{22}(t) & \cdots & \sigma_{2d}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{n1}(t) & \sigma_{n2}(t) & \cdots & \sigma_{nd}(t)
\end{bmatrix}
\]

\( -n \times m \) matrix

The equation is

\[ dX(t) = \mu(t) dt + \sigma(t) dW(t) \]
Let $f \in C^2(\mathbb{R}^n)$. Seek to calculate $f(X(t)) := z(t)$.

We could use Taylor’s formula to derive the following Ito formula:

$$
\frac{df(X(t))}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(X(t)) dX_i(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial X_i \partial X_k}(X(t)) dX_i(t) dX_k(t)
$$

When you compute the products $dX_i(t)dX_k(t)$ keep in mind that if $i \neq k$ then $W_i(t)$ and $W_k(t)$ are independent Bms and that translates into $dW_i(t)dW_k(t) = 0$, (if $W_i$ and $W_k$ are independent so all their increments.)

Hence,

$$
\frac{df(X(t))}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(X(t)) \mu_i(t) dt + \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(X(t)) \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t)
$$

In matrix notation:

$$
\frac{df(X(t))}{dt} = (\nabla f)(X(t)) \cdot (\mu(t) dt) + (\nabla f)(X(t)) \cdot (\sigma(t) dW(t)) + \frac{1}{2} (\nabla^2 f)(X(t)) \cdot (\sigma^*(t) dt)
$$

The last term can also be written as:

$$
\frac{1}{2} \text{trace}[(\sigma^*)^2(t) \nabla^2 f(X(t))] dt
$$

**Integration by parts formula**

Let $X_1(t)$ and $X_2(t)$ be semimartingales as in the previous notation with $n = 2, d = 2$.

**Theorem:**

$$
X_1(t)X_2(t) = X_1(0)X_2(0) + \int_0^t X_1(s) dX_2(s) + \int_0^t X_2(s) dX_1(s) + \int_0^t <X_1;X_2>_s ds
$$

**Ex**

Under which conditions the process defined by

$$
X(t) = f(W_1(t), W_2(t))
$$

where $W_1, W_2$ are two independent standard Bms is a martingale?

**Stochastic differential equations**

Consider the following

\[
\begin{cases}
    dX(t) = \mu(t, X(t)) dt + \sigma(t, X(t)) dW(t) \\
    X(0) = x
\end{cases}
\]
where $\mu$ and $\sigma$ are such that:

1. $\forall x$: $\mu(\cdot, x), \sigma(\cdot, x)$ are stochastic processes, adapted.

2. $\exists$ a constant $L$ such that for $\forall t$ and any $x, y$
   
   $|\mu(t, x) - \mu(t, y)| \leq L|x - y|$
   
   $|\sigma(t, x) - \sigma(t, y)| \leq L|x - y|$

3. $\exists$ a constant $M$ such that for $\forall t$ and any $x, y$
   
   $|\mu(t, x)| \leq M(1 + |x|)$
   
   $|\sigma(t, x)| \leq M(1 + |x|)$

**Theorem:**

Under these conditions $*$ has a unique solution such that for any $t$ and any $x$, $\int_0^t E[X(s)^2]ds < \infty$