Lecture 11

Stochastic differential equations

The concept of SDE was introduced last semester, using Ito formula to formalize it. The notation

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t, \quad t \in [0, \infty)$$

was justified as a symbolic way of writing

$$\int_0^t dS_s = \int_0^t \mu(S_s, s)ds + \int_0^s \sigma(S_s, s)dW_s,$$

when $t$ is infinitesimal.

Remark: Obviously, the SDE above depends a lot on the set of information $\mathcal{F}_t$ to which $W$ is adapted. For example, if a market participant has inside information and learns all the random events that influence price changes in advance, the diffusion term would be zero, since the participant knows how $dS_t$ is going to change, he can predict this perfectly, and $dW_t = 0$. If we were to write this participant’s SDE we would get

$$dS_t = \mu^*(S_t, y)dt$$

but for all the other market participants

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t, \quad t \in [0, \infty)$$

Remark: The parameter $\mu$ and $\sigma$ are allowed to depend on $S_t$ and $t$, therefore are themselves random variables. They are also adapted to the same set of information $\mathcal{F}_t$. We decided last semester that in order for the SDE to make sense they have to verify some regular conditions. They are

1) $\mu(S_t, t) \in L^1(\Omega)$, i.e. $E(|\mu(S_t, t)|) < \infty$,
2) $\sigma(S_t, t) \in L^2(\Omega)$, i.e. $E(\sigma^2(S_t, t)) < \infty$.

These conditions have similar meanings. They require that the drift and diffusion do not vary "too much" over time.

Solutions

The unknown in a SDE is a stochastic process. Therefore the notion of a solution for a SDE is a bit more complicated, because we are not searching for a number, or vector, but for a sp whose trajectories and probabilities associated to those trajectories need to be determined exactly. In stochastic calculus we are talking about two types of solutions.

Strong solutions These are similar to the case of ODE. Given the drift, diffusion parameters, and the random innovation term $dW_t$, we determine a random process $S_t$, whose path satisfy our SDE. Clearly such a process will depend on time $t$, and on the past and contemporaneous values of the random variable $W_t$, as the underlying integral equation illustrates:
\[
\int_0^t dS_s = \int_0^t \mu(S_s, s) ds + \int_0^s \sigma(S_s, s) dW_s,
\]
for all \( t > 0 \). The solution determines the exact form of this dependence. When \( W_t \) is given exogenously and \( S_t \) is then determined, we obtain the so-called strong solution of the SDE.

**Weak solution** The second solution concept is specific to SDEs. It is called the weak solution. In the weak solution, one determines the process \( \tilde{S}_t \),

\[
\tilde{S}_t = f(t, \tilde{W}_t)
\]

where \( \tilde{W}_t \) is a Wiener process whose distribution is determined simultaneously with \( \tilde{S}_t \). For the weak solution of an SDE we just need \( \mu(\cdot) \) and \( \sigma(\cdot) \). So here we have to find the pair \( \tilde{S}_t, \tilde{W}_t \) that satisfies the equation. Here \( \tilde{W}_t \) is not “given” but needs to be determined.

BUT, what is the difference between \( W_t \) and \( \tilde{W}_t \) if both are Bm? Are these not the same objects? Statistically they are the same, i.e., you can not distinguish them by looking at the mean, variance, etc. BUT they are adapted to different sets of information! Even if they have the same statistical properties the two processes could represent very different real-life phenomena if they are measurable with respect to different information sets.

In conclusion, the strong solution calculates \( S_t \) with \( dW_t \) given, that is, in order to obtain it we need to know the family of information \( \mathcal{F}_t \). This means that the strong solution is \( \mathcal{F}_t \) adapted. The weak solution, on the other hand, is not calculated using the process that generates the set of information \( \mathcal{F}_t \). Instead, it is found along with some process \( \tilde{W}_t \). The process \( \tilde{W}_t \) could generate other set of information \( \mathcal{H}_t \). The corresponding \( \tilde{S}_t \) will not necessarily be \( \mathcal{F}_t \) adapted, but \( \tilde{W}_t \) is a Bm, hence a martingale, with respect to \( \mathcal{H}_t \).

The use of a strong solution implies knowledge of the error process \( W_t \). If this is the case, the financial analyst may work with strong solutions. But often when the price of a derivative is calculated using a solution to an SDE, one does not know the exact process \( W_t \). One may use only the volatility and (sometimes) the drift component. Hence, in pricing derivative products under such conditions, one works with weak solutions.

**Theorem:**

Consider the following

\[
\begin{cases}
  dX(t) = \mu(t, X(t)) dt + \sigma(t, X(t)) dW(t) \\
  X(0) = x 
\end{cases}
\]

where \( \mu \) and \( \sigma \) are such that:

1. \( \forall x: \mu(\cdot, x), \sigma(\cdot, x) \) are stochastic processes, adapted.
2. \exists \text{ a constant } L \text{ such that for } \forall t \text{ and any } x, y
\begin{align*}
& |\mu(t, x) - \mu(t, y)| \leq L|x - y| \\
& |\sigma(t, x) - \sigma(t, y)| \leq L|x - y|
\end{align*}

3. \exists \text{ a constant } M \text{ such that for } \forall t \text{ and any } x, y
\begin{align*}
& |\mu(t, x)| \leq M(1 + |x|) \\
& |\sigma(t, x)| \leq M(1 + |x|)
\end{align*}

Under these conditions * has a unique solution such that for any \( t \) and any \( x \),
\[
\int_0^t E[X(s)^2]ds < \infty
\]

**Examples:**

(1) *Geometric Brownian motion.* Let's look at the following SDE:
\[
dX(t) = \alpha X(t)dt + \sigma X(t)dW(t).
\]

This is the model used by Black and Scholes. The drift and the diffusion coefficients depend on the information that becomes available at time \( t \). They change proportional with \( S_t \). This means that although the drift and the diffusion part of the increment in the asset price changes, the drift and the diffusion of percentage change in \( S_t \) has time invariant parameters.

By the previous theorem there is a solution for this SDE. Why?

One could check that the unique solution is:
\[
X(t) = X(0) \exp \left( \left( \alpha - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right).
\]

It is called *geometric Brownian motion.*

**Properties:**
1) If \( \alpha = 0 \) then \( X \) is a martingale.
2) It is positive!!
3) \( E[X(t)] = e^{\alpha t} \)

We observe in the picture that there is an exponential trend, and random fluctuations around this trend. These variations increase over time because of higher prices. For more asset prices an exponential trend is more realistic than a linear trend.

(2) *Ornstein–Uhlenbeck processes.* These are used in interest rate theory.

There are not many SDEs that we could solve, but the ones we solve we usually get inspiration from the ODE case and then "guess" the solution. Of course, we have to check that our guess is the right one.

Look at the following SDE:
\[
dX(t) = \alpha X(t)dt + \sigma dW(t),
\]
\[ X(0) = x_0. \]

Here \( \sigma \) and \( \alpha \) are constants. We know from the previous theorem that there is a solution. What are \( L \) and \( M \)? How do we find the solution?

Inspiration: Let’s look at the non-stochastic differential equation

\[ dX(t) = \alpha x(t)dt + \sigma \omega(t)dt \]

Solution:

\[ e^{\alpha t} \left( x(0) + \int_0^t e^{-\alpha s} \omega(s)ds \right). \]

This is a product of 2 functions!! Let’s try to guess our solution:

Let

\[ X(t) = e^{\alpha t} \left( x_0 + \sigma \int_0^t e^{-\alpha s} dW(s) \right) \]

Now we can apply the product rule:

Do you already know this process?

(a) Vasicek interest rate model:

\[ Dr = (b - AR)dt + \sigma dW \]

Mean-reverting model.

(b) Autregressive of time series and Longevin’s SDE.