Lecture 2

Pricing an option: a discrete model

Model specification (simple period model)

We fix the following notations:

1) There are 2 times (dates) \( t = 0 \) and \( t = 1 \). We could trade (or consume) at these 2 dates.

2) There are \( k \) possible states of the world, the value of which is unknown at time \( t = 0 \), but it is known at time \( t = 1 \). We write this in a sample space.

\[ \Omega = \{ \omega_1, \omega_2, ..., \omega_k \} \]

3) A probability measure \( P \) on \( \Omega \), with \( P(\omega) > 0 \) for all \( \omega \in \Omega \) and \( \sum_{i=1}^{k} P(\omega_i) = 1 \) is given.

4) A price process \( S = \{ S_t, t = 0, 1 \} \) where \( S_t = (S_1(t), S_2(t), ..., S_N(t)) \), \( N < \infty \) and \( S_n(t) \) is the time \( t \) price of security \( n \).

Remarks:

a) These risky securities are most of the time stocks.

b) At time \( t = 0 \) the prices are known to the investor and they are positive scalars.

c) At time \( t = 1 \) the prices are non-negative random variables whose values become known to the investor only at time \( t = 1 \).

5) There is a bank account process \( B = \{ B_t, t = 0, 1 \} \) where \( B_0 = 1 \) and \( B_1 \) is a random variable. For the bank account \( B_1(\omega) > 0 \) for all \( \omega \)'s. This is different from the risky securities where \( S_1(\omega) \) could be 0. Usually, in fact, \( B_1 \geq 1 \), but it is not necessary. We could look at \( B_1 \) as being the time \( t = 1 \) value of a bank account of $1 at time \( t = 0 \). Then the interest rate is \( r = B_1 - 1 \) and there are models where we allow \( r \leq 0 \). Nevertheless \( B_1 > 0 \).

Definition

A trading strategy, denoted by \( H = (H_0, H_1, ..., H_N) \) describes an investor’s portfolio as carried forward from time \( t = 0 \) to time \( t = 1 \).

\( H_0 \) = # of dollars invested in the bank.

\( H_n \) = # of units of security \( n \) held between \( t = 0 \) and \( t = 1 \).

Remarks:

a) \( H_n \) could be negative, or positive.

b) If \( H_n \) is negative that means that we borrow (if \( n = 0 \)) or we are selling short, or have a short position in asset \( n \). If \( H_n \) is positive then we say we have a long position in asset \( n \).
**Definition**

The value process $V = \{V_t, t = 0, 1\}$ is given by the total value of the portfolio at each point in time, i.e.

$$V_t = H_0 B_t + \sum_{n=1}^{N} H_n S_n(t), (t = 0, 1)$$

or $V_0 = H_0 B_0 + \sum_{n=1}^{N} H_n S_n(0)$

$$V_1 = H_0 B_1 + \sum_{n=1}^{N} H_n S_n(1)$$

$$V_1 - V_0 = H_0 (B_1 - B_0) + \sum_{n=1}^{N} H_n (S_n(1) - S_n(0))$$

$$V_1 - V_0 = H_0 r + \sum_{n=1}^{N} H_n \Delta S_n$$

This quantity is called the gain process $G = V_1 - V_0$ and it is a random variable. $G$ is the total profit or loss generated by the portfolio between times $t = 0$ and 1

(We do not include addition of funds or consumption!)

**Definition**

The discounted price process $S^* = \{S_t^*, t = 0, 1\}$, $S_t^* = (S_1^*(t), ..., S_N^*(t))$ is defined by

$$S_n^*(t) = \frac{S_n(t)}{B_t} \quad n = 1, ..., N, \quad t = 0, 1$$

We normalize the prices such that the bank becomes constant (is the numeraire)

So

$$V_t^* = H_0 + \sum_{n=1}^{N} H_n S_n^*(t)$$

and the discounted gains process:

$$G^* = \sum_{n=1}^{N} \Delta S_n^*$$

Observe that $V_t^* B_t = V_t \implies V_t^* = \frac{V_t}{B_t}$ and $G^* = V_1^* - V_0^*$. **Arbitrage**

**Definition**
An opportunity of making a profit on a transaction without being exposed to the risk of incurring a loss is called an arbitrage opportunity. Formally, an arbitrage opportunity is some strategy \( H \) such that
\[
(a) \quad V_0 = 0 \\
(b) \quad V_1 \geq 0 \\
(c) \quad EV_1 > 0
\]

**Remark**

1) This is a riskless way of making money. You start with 0 money and without a chance of going into debt (debt means \( V_1 < 0 \)). There is a chance of ending up with a positive amount of money (\( EV_1 > 0 \)).

2) An economical model with arbitrage opportunities will not be in equilibrium, therefore we are interested in models (that are interesting from economic standpoint) that don’t have arbitrage opportunities. The question that we will explore is: what are the conditions that have to be satisfied in order for the model to be free of arbitrage. There is no easy way to check directly whether a model has any arbitrage opportunities, but there are necessary and sufficient conditions for the model to be free of arbitrage.

3) \( H \) is an arbitrage opportunity if and only if:
\[
(a) \quad G^* \geq 0 \\
(b) \quad EG^* > 0 \\
(c) \quad V_0^* = 0
\]

To see this, consider \( H – \) an arbitrage opportunity. we saw
\[
G^* = V_1^* - V_0^* = V_1^* - 0 \geq 0 \\
EG^* = EV_1^* - EV_0^* = EV_1^* - 0 = EV_1^* > 0
\]

So
\[
G^* \geq 0, \quad EG^* > 0, \quad V_0^* = 0.
\]

Conversely, suppose there is a strategy \( \hat{H} \) such that \( G^* \geq 0 \) and \( EG^* > 0 \). Consider the strategy
\[
H = (H_0, \hat{H}_1, ..., \hat{H}_N) \quad \text{where} \quad H_0 = -\sum_{n=1}^{N} \hat{H}_n S_n^*(0).
\]

Then \( V_0^* = 0 \) and \( V_1^* = V_0^* + G^* = G^* \geq 0 \). Also, \( EV_1^* = EG^* > 0 \). So \( H \) is an arbitrage opportunity.

**Definition**

It is said that the law of one price holds for a securities market model if there do not exist two trading strategies, say \( \tilde{H} \) and \( \tilde{H} \), such that \( \tilde{V}_1(\omega) = \tilde{V}_1(\omega) \) for all \( \omega \in \Omega \), but \( \tilde{V}_0 > \tilde{V}_0 \). This means that the law of one price holds if it is not possible to find two trading strategies that give the same payoff at time \( t = 1 \) but the initial values of the two corresponding portfolios are different.

**Definition**
A trading strategy $\hat{H}$ is said to be dominant if there exists another trading strategy, say $\tilde{H}$, such that $V_0 = \tilde{V}_0$ and $\hat{V}_1(\omega) > \tilde{V}_1(\omega)$ for all $\omega \in \Omega$. In other words, both trading strategies start with the same amount of money, but the dominant one is certain to end up with more.

**Proposition**

There exists a dominant trading strategy iff there is a strategy $V_0 = 0$ and $V_1(\omega) > 0$ for all $\omega \in \Omega$.

**Remark:** If there is a dominant trading strategy, then there exists a trading strategy that can transform a strictly negative initial wealth into a non-negative wealth. SHOW THIS AS A HOMEWORK EXERCISE.

**Proposition**

If there are no dominant trading strategies, then the law of one price holds. The converse is not true.

The pricing of claims will be logically consistent if there is a linear pricing measure, i.e. a non-negative vector $\pi = (\pi(\omega_1), ..., \pi(\omega_n))$ such that for every trading strategy $H$ you have

$$V^*_0 = \sum_{\omega} \pi(\omega)V^*_1(\omega) = \sum_{\omega} \pi(\omega) \frac{V_1(\omega)}{B_1(\omega)}$$

Each claim has a unique price, and a claim that pays more than another in every state will have a higher time $t = 0$ price.

**Proposition**

The vector $\pi$ is a linear pricing measure if and only if it is a probability measure on $\Omega$ satisfying

$$S^*_n(0) = \sum_{\omega} \pi(\omega)S^*_n(1)(\omega) \quad n = 1, 2, ..., N$$

(This means that the expectation under $\pi$ of the final discounted price equals the initial price of each security.)

**Proposition**

There exists a linear pricing measure if and only if there is no dominant strategy.