Lecture 3

We saw, an arbitrage opportunity is some strategy $H$ such that
(a) $V_0 = 0$
(b) $V_1 \geq 0$
(c) $EV_1 > 0$

OR

(a) $G^* \geq 0$
(b) $EG^* > 0$
(c) $V_0^* = 0$

It is said that the law of one price holds for a securities market model if there do not exist two trading strategies, say $\hat{H}$ and $\tilde{H}$, such that $\tilde{V}_1(\omega) = \hat{V}_1(\omega)$ for all $\omega \in \Omega$, but $\hat{V}_0 > \tilde{V}_0$.

A trading strategy $\hat{H}$ is said to be dominant if there exists another trading strategy, say $\tilde{H}$, such that $\tilde{V}_0 = \hat{V}_0$ and $\tilde{V}_1(\omega) > \hat{V}_1(\omega)$ for all $\omega \in \Omega$. **Proposition** There exists a dominant trading strategy iff there is a strategy $V_0 = 0$ and $V_1(\omega) > 0$ for all $\omega \in \Omega$. And if there is a dominant trading strategy, then there exists a trading strategy that can transform a strictly negative initial wealth into a non-negative wealth.

Now, lets see why:

**Proposition** If there are no dominant trading strategies, then the law of one price holds. The converse is not true.

The pricing of claims will be logically consistent if there is a linear pricing measure, i.e. a non-negative vector $\pi = (\pi(\omega_1), ..., \pi(\omega_n))$ such that for every trading strategy $H$ you have

$$V_0^* = \sum_\omega \pi(\omega)V_1^*(\omega) = \sum_\omega \pi(\omega) \frac{V_1(\omega)}{B_1(\omega)}$$

Each claim has a unique price, and a claim that pays more than another in every state will have a higher time $t = 0$ price.

**Proposition**

The vector $\pi$ is a linear pricing measure if and only if it is a probability measure on $\Omega$ satisfying

$$S_n^*(0) = \sum_\omega \pi(\omega)S_n^*(1)(\omega) \quad n = 1, 2, ..., N$$

(This means that the expectation under $\pi$ of the final discounted price equals the initial price of each security.)

**Proposition**

There exists a linear pricing measure if and only if there is no dominant strategy.

**Risk neutral probabilities**
We saw that a linear pricing measure was a probability with a certain property. Its existence implied no dominant trading strategies but arbitrage could still exist. In order for the arbitrage to be ruled out we need to add on extra condition: the linear pricing measure needs to give strictly positive mass to every state $\omega \in \Omega$.

**Definition**

A probability $Q$ on $\Omega$ is said to be a risk neutral probability measure if

(a) $Q(\omega) > 0$ for all $\omega \in \Omega$.
(b) $E_Q(\Delta S^*_n) = 0$, $n = 1, 2, ..., N$

where $E_Q$-means expectation under probability $Q$. ($E_Q(\Delta S^*_n) := \sum_{\omega} S^*_n(\omega)Q(\omega)$)

**Remark**

$E_Q(\Delta S^*_n) = E_Q[S^*_n(1) - S^*_n(0)] = E_Q[S^*_n(1)] - E_Q[S^*_n(0)] = E_Q[S^*_n(1)] - S^*_n(0)$

Hence for the risk neutral probability we have $E_Q(S^*_n(1)) = S^*_n(0)$

**FIRST FUNDAMENTAL THEOREM OF FINANCE**

There are no arbitrage opportunities if and only if there exists a risk neutral probability measure $Q$.

**Valueation of a contingent claim**

**Definition** A contingent claim $X$ is said to be attainable (or marketable, or replicable, or reachable) if there exists some trading strategy $H$, called the replicating portfolio (hedging portfolio) such that $V_1 = X$. We say that $H$ generates $X$.

**Remark**

Denote by $p$ the price of $X$ at time $t = 0$

If $p > V_0$, then one could make a riskless investment, making $p - V_0$. How?

If $p < V_0$, he’ll make $V_0 - p$ risk free.

If $p = V_0$, then we can not use $H$ to create a profit? Does this mean that $V_0$ is the correct value of $X$? Not necessary if the law of one price does not hold.

**Proposition** If $Q$ is any risk-neutral probability measure, then for every trading strategy $H$ one has:

$$V_0 = E_Q[V_1/B_1] = E_Q[V^*_1]$$

**Remark**

1) You can not have two trading strategies $H, \hat{H}$, such that $V_1 = \hat{V}_1$ but $V_0 = \hat{V}_0$ if there is a risk-neutral probability.
2) The equality $V_0 = E_Q[V^*_T]$ does not depend on the choice of $Q$. So if there are many risk-neutral probabilities, the quantity $E_Q[V^*_T]$ is constant.

**Valuation concept**

If the law of one price holds, then the time $t = 0$ value of an attainable contingent claim $X$ is $V_0 = H_0B_0 + \sum_{n=1}^N H_nS_n(0)$ where $H$ is the trading strategy that generates $X$.

**Risk-neutral valuation principle**

If the single period model is free of arbitrage opportunities, then the time $t = 0$ value of an attainable contingent claim $X$ is $E_Q[X/B_1]$ where $Q$ is any risk-neutral probability measure.