Lecture 5

Multiperiod security markets

Many practitioners are using multiperiod models (rather than continuous models).

Settings:
1) There are $T + 1$ trading dates $t = 0, 1, ..., T$
2) There are a finite number ($k$) of states of the world. $\Omega = \{\omega_1, \omega_2, ..., \omega_k\}$
3) There is a probability measure $P$ on $\Omega$, with $P(\omega) > 0$ for all $\omega \in \Omega$. $P(\omega_i)$ represents the probability that the state $i$ occurs.
4) A filtration $(\mathcal{F}_t)_{t=0,1,...,T}$ that contains all the relevant information up to time $t$ about the security prices.
5) A bank account process $B = \{B_t; t = 0, 1, ..., T\}$ where $B$ is a stochastic process (discrete) with $B_0 = 1$ and $B_t(\omega) > 0$ for all $t$ and $\omega$. $B_t$ is the time $t$ value of a $1$ deposited at time $t = 0$. Usually $B$ is a non-decreasing process, and the (possibly random) quantity $r_t = (B_t - B_{t-1})/B_{t-1} \geq 0$ $t = 1, ..., T$, should be thought of as interest rate pertaining to the time interval $(t-1, t)$.
6) $N$ risky securities processes $S_n = \{S_n(t); t = 0, 1, ..., T\} n = 1, 2, ..., N$ with $S_n$ being a nonnegative stochastic process for each $n = 1, 2, ..., N$. Here $S_n(t)$ is the stock price at time $t$ of the risky security.

Differences from the one period model

— We have $T + 1$ trading dates.
— We have a filtration.
— We deal with discrete stochastic processes (rather than just random variables.)
— Keep in mind, we have $k$ possible states of the world at time $T$. The way we get there is dictated by the filtration $\mathcal{F}$.

Value Process and Gains Process

A trading strategy $H = (H_0, H_1, ..., H_N)$ is a vector of stochastic processes $H_n = \{H_n(t); t = 1, 2, ..., T\}, n = 0, 1, ..., N$. Here $H_n(t)$ is the number of shares (units) that the investor owns from time $t - 1$ to time $t$. $H_0(t)B_{t-1}$ equals the amount of money invested in the bank at time $t - 1$. Remark that $H_n(t)$ can be negative.

Definition
The value process \( V = \{ V_t; t = 0, 1, \ldots, T \} \) is a stochastic process defined by:

\[
V_t = \begin{cases} 
H_0(1)B_0 + \sum_{n=1}^{N} H_n(1)S_n(0), & t = 0 \\
H_0(t)B_t + \sum_{n=1}^{N} H_n(t)S_n(t), & t \geq 1
\end{cases}
\]

\( V_0 \) — is the time \( t = 0 \) value of the portfolio.
\( V_t \) — is the time \( t \) value of the portfolio before any transaction is made at time \( t \).

**Notation**

\( \Delta S_n(t) = S_n(t) - S_n(t - 1) \) — the change in value of the stock price between times \( t - 1 \) and \( t \). Then \( H_n(t)\Delta S_n(t) \) is the one-period gain or loss between \( t - 1 \) and \( t \) for the security \( n \).

**Definition**

\[
G_t = \sum_{s=1}^{t} H_0(s)\Delta B_s + \sum_{n=1}^{N} \sum_{s=1}^{t} H_n(s)\Delta S_n(s), t \geq 1
\]

defines the gains process — represents the cumulation gain or loss up to time \( t \) of the portfolio.

**Remark**

This looks like a stochastic integral. It is the stochastic integral of the trading strategy with respect to the price process.

**Remark**

The time \( t \) value of the portfolio, just after any time \( t \) transaction is made is:

\[
H_0(t + 1)B_t + \sum_{n=1}^{N} H_n(t + 1)S_n(t), \quad t \geq 1
\]

In general \( V_t \) and this could be different due to inflows or outflows (transaction costs, consumption) of money from the portfolio.

**Definition**

A trading strategy is said to be self-financing if

\[
V_t = H_0(t + 1)B_t + \sum_{n=1}^{N} H_n(t + 1)S_n(t), \quad t = 1, 2, \ldots, T - 1
\]

(Any change in the value of the portfolio is due to gains or loss in the investment but not because of addition or withdrawal of money.)

**Discounted prices**

Most of the time we are not interested in absolute value of the securities but rather discounted ones.
**Definition**

The discounted price process

\[ S_n^*(t) = \frac{S_n(t)}{B_t}, \quad t = 0, 1, ..., T \quad n = 1, 2, ..., N \]

discounted value process

\[ V^* = \{V^*_t\}, \quad V^*_t = \frac{V_t}{B_t}, \quad t = 0, ..., T \]

\[ V^*_t = \begin{cases} H_0(1) + \sum_{n=1}^{N} H_n(1)S_n^*(0), & t = 0 \\ H_0(t) + \sum_{n=1}^{N} H_n(t)S_n^*(t), & t = 1, T \end{cases} \]

discounted gains process

\[ G^* = \sum_{n=1}^{N} \sum_{u=1}^{t} H_n(u) \Delta S_n^*(u), \quad t = 1, T, \quad V^*_t = V_0^* + G_t^* . \]

**Arbitrage in multiperiod models**

In multiperiod models arbitrage is defined similarly.

**Definition**

1) An arbitrage opportunity is some trading strategy \( H \) such that:

(a) \( V_0 = 0 \)
(b) \( V_T \geq 0 \)
(c) \( EV_T > 0 \)
(d) \( H \) is self-financing

2) A self financing strategy \( H \) is an arbitrage opportunity iff:

(a) \( V_0^* = 0 \) or (a) \( G_T^* \geq 0 \)
(b) \( V_T^* \geq 0 \) or (b) \( EG_T^* > 0 \)
(c) \( EV_T^* > 0 \) or (c) \( V_0^* = 0 \)

But the risk neutral measure the definition is a bit different:

**Definition**

A risk neutral measure (martingale measure) is a probability measure \( Q \) such that:
1) \( Q(\omega) > 0 \) for all \( \omega \in \Omega \)

2) \( S_n^* \) is a martingale under \( Q \) for every \( n = 1, 2, ..., N \)

\[
E_Q[S_n^*(t + s)|\mathcal{F}_t] = S_n^*(t), \quad t, s \geq 0
\]

or

\[
E_Q[S_n^*(t + s) - S_n^*(t)|\mathcal{F}_t] = 0
\]

or

\[
E_Q[S_n(t + s)/B(t + s)|\mathcal{F}_t] = S_n(t)/B(t)
\]

\[
\implies E_Q[B_t S_n(t + s)/B(t + s)|\mathcal{F}_t].
\]

**First fundamental theorem of finance:**

There are no arbitrage opportunities if and only if there exists a martingale measure \( Q \).

**Proposition**

If the multiperiod model does not have any arbitrage opportunity, then none of the underlying single period models has any arbitrage opportunities in the single period sense.

**Example**

Consider a 2-period problem with \( \Omega = \{\omega_1, \omega_2, ..., \omega_5\} \) and one risky security:

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( S_0(\omega) )</th>
<th>( S_1(\omega) )</th>
<th>( S_2(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 )</td>
<td>6</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>( \omega_3 )</td>
<td>6</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>( \omega_4 )</td>
<td>6</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>( \omega_5 )</td>
<td>6</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

Find: \( V_t, V_t^*, G_t^*, G_t \) for all strategies \( (H_0, H_1) \) and check for the existance of martingale measures or linear pricing measure. Find all of them. Keep in mind \( B_t = (10/9)^t \).

Is the strategy: \( H_0(1)(\omega) = 2 \) for all \( \omega \)'s a self financing startegy?

\[
H_1(1)(\omega) = 3 \quad \text{for all} \quad \omega
\]

\[
H_0(2)(\omega) = \begin{cases} 
1, & \text{for} \quad \omega = \omega_1, \omega_2, \omega_3 \\
2, & \text{for} \quad \omega = \omega_4, \omega_5 
\end{cases}
\]

\[
H_1(2)(\omega) = \begin{cases} 
29/9, & \text{for} \quad \omega = \omega_1, \omega_2, \omega_3 \\
4, & \text{for} \quad \omega = \omega_4, \omega_5 
\end{cases}
\]