Arbitrage in multiperiod models

In multiperiod models arbitrage is defined similarly.

**Definition**

1) An arbitrage opportunity is some trading strategy $H$ such that:
   
   (a) $V_0 = 0$
   
   (b) $V_T \geq 0$
   
   (c) $EV_T > 0$
   
   (d) $H$ is self-financing

2) A self financing strategy $H$ is an arbitrage opportunity iff:
   
   (a) $V_0^* = 0$ or (a) $G_T^* \geq 0$

   (b) $V_T^* \geq 0$ or (b) $EG_T^* > 0$

   (c) $EV_T^* > 0$ or (c) $V_0^* = 0$

But the risk neutral measure the definition is a bit different:

**Definition**

A risk neutral measure (martingale measure) is a probability measure $Q$ such that:

1) $Q(\omega) > 0$ for all $\omega \in \Omega$

2) $S_n^*$ is a martingale under $Q$ for every $n = 1, 2, ..., N$

   $$E_Q[S_n^*(t + s)|\mathcal{F}_t] = S_n^*(t), \quad t, s \geq 0$$

or

   $$E_Q[S_n^*(t + s) - S_n^*(t)|\mathcal{F}_t] = 0$$

or

   $$E_Q[S_n(t + s)/B(t + s)|\mathcal{F}_t] = S_n(t)/B(t)$$

   $$\implies E_Q[B_t S_n(t + s)/B(t + s)|\mathcal{F}_t].$$

**First fundamental theorem of finance:**

There are no arbitrage opportunities if and only if there exists a martingale measure $Q$.

**Proposition**
If the multiperiod model does not have any arbitrage opportunity, then none of the underlying single period models has any arbitrage opportunities in the single period sense.

Example

Consider a 2-period problem with $\Omega = \{\omega_1, \omega_2, ..., \omega_5\}$ and one risky security:

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$S_0(\omega)$</th>
<th>$S_1(\omega)$</th>
<th>$S_2(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>6</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>6</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>6</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>6</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

Find: $V_t$, $V^*_t$, $G_t$, $G^*_t$ for all strategies $(H_0, H_1)$ and check for the existence of martingale measures or linear pricing measure. Find all of them. Keep in mind $B_t = (\frac{10}{9})^t$.

Is the strategy: $H_0(1)(\omega) = 2$ for all $\omega$'s a self financing strategy?

$H_1(1)(\omega) = 3$ for all $\omega$'s

$H_0(2)(\omega) = \begin{cases} 
1, & \text{for } \omega = \omega_1, \omega_2, \omega_3 \\
2, & \text{for } \omega = \omega_4, \omega_5
\end{cases}$

$H_1(2)(\omega) = \begin{cases} 
29/9, & \text{for } \omega = \omega_1, \omega_2, \omega_3 \\
4, & \text{for } \omega = \omega_4, \omega_5
\end{cases}$

The binomial model

Is a particular case of the multiperiod model. Each period there are 2 possibilities: the security price either goes up by the factor $u$ ($u > 1$) or goes down by a factor $d$ ($0 < d < 1$). The probability of an up move is equal to $p$, and the moves over time are independent of each other. Hence the binomial model is related to the Binomial process (Bernoulli process) from probability in the following fashion.

$$S_{n+1} = S_n Z_n$$

where $\{Z_n\}_{n \in \mathbb{N}}$ are iid with the distribution:

$$z = \begin{cases} 
u, & \text{with probability } p \\
d, & \text{with probability } 1 - p
\end{cases}$$

This is a multiplicative model. The reason we prefer multiplicative models to additive ones is because stock prices have an exponential behavior that could be explained by the multiplicative models. Also, additive models may result in negative prices.
Let $k = \#$ of up moves. in general, if

$$\begin{align*}
\# \text{ up steps} &= n \\
\text{total } \# \text{ steps} &= t \quad t \geq n
\end{align*}$$

then the $\#$ of path to reach state “m” = \binom{t}{m}

$$P(S_t = S u^n d^{t-n}) = \binom{t}{m} p^n (1-p)^{t-n} \quad n = 0, 1, ..., t$$

The self financing equation now is

$$H_0(t)(1 + R) + H_1(t)S_t = H_0(t+1) + H_1(t+1)S_t$$

What equation does the risk neutral probability verify here?

$$S_0 = \frac{1}{1+r} E^Q[S_1|\mathcal{F}_0], \quad \text{in general} \quad S_t = \frac{1}{1+r} E^Q[S_{t+1}|\mathcal{F}_t]$$

$$q \left[ \frac{u-1-r}{1+r} \right] + (1-q) \left[ \frac{d-1-r}{1+r} \right] = 0 \quad q = \frac{1+r-d}{u-d}$$

In general $q$ is the conditional probability the next move is an “up” move given the information $\mathcal{F}_t$ at time $t$. So $q = \frac{1+r-d}{u-d}$ for all $\mathcal{F}_t$ and $t$.

Also remark that in order for $Q$ to exist we need $0 < q < 1 \rightarrow d < 1 + r < u$

The martingale measure will be given by

$$Q(\omega) = q^n (1-q)^{T-n}$$

where $\omega$ is any state corresponding to “n” ups and “T-n” downs.

Hence the probability distribution of $S_t$ under the risk neutral probability measure is given for all $t$ by

$$Q(S_t = S_0 u^n d^{t-n}) = \binom{t}{m} q^n (1-q)^{t-n}, \quad n = 0, 1, ..., t.$$ 

A contingent claim is a r.v. $X$ that represents the time $T$ pay off from a “seller” to a “buyer”. The contingent claim is simple if $X = \Phi(S(T))$.

**Example**

European call option $X = max\{S(T)-K, 0\}$. If

<table>
<thead>
<tr>
<th>$\omega_k$</th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>$S_0 = 5$</td>
<td>$S_1 = 8$</td>
<td>$S_2 = 9$</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>$S_0 = 5$</td>
<td>$S_1 = 8$</td>
<td>$S_2 = 6$</td>
</tr>
</tbody>
</table>
\( \omega_3 \quad s_0 = 5 \quad s_1 = 4 \quad s_2 = 6 \)
\( \omega_4 \quad s_0 = 5 \quad s_1 = 4 \quad s_2 = 3 \)

then \( X = \max \{s_2 - K, 0\} \), for \( K = 5 \)
\[
X(\omega) = \begin{cases} 
4, & \omega = \omega_1 \\
1, & \omega = \omega_2, \omega_3 \\
0, & \omega = \omega_4 
\end{cases}
\]

Example of a contingent claim (derivative security) that is not simple:
\[
X = \max \left\{ \frac{s_0 + s_1 + s_2}{3 - 5}, 0 \right\}
\]
This is called an Asian or averaging option.
\[
X(\omega) = \begin{cases} 
7/3, & \omega = \omega_1 \\
4/3, & \omega = \omega_2 \\
0, & \omega = \omega_3, \omega_4 
\end{cases}
\]

Question:
The question of pricing is similar to the one-period model. Find \( \pi(t, x) \) – the price of the contingent claim at time \( t = 0, 1, ..., T \).

Definition
A contingent claim is said to be marketable or attainable if there exists a self-financing strategy such that \( V_T(\omega) = X(\omega) \) for all \( \omega \in \Omega \). The corresponding portfolio or trading strategy \( H \) is said to be replicate or generate \( X \).

Risk neutral valuation principle
The time \( t \) value of a marketable contingent claim \( X \) is equal to \( V_t \), the time \( t \) of the portfolio which replicates \( X \). Moreover:
\[
V^*_t = V_t / B_t = E_Q[X / B_T | {\mathcal{F}_t}], \quad t = 0, 1, ..., T
\]
for all risk neutral probability measures \( Q \).

Binomial model review
Recall that at time \( T \), each possibility for \( S_T \) is parameterized by \( K = \# \) of ups from \( t = 0 \) to \( T \).

So the value of the option at time \( T \), if \# ups = \( k \)
\[
V_T(k) = \Phi(s_0u^k d^{T-k})
\]

We saw that
\[
V_0 = \frac{1}{(1+r)^T} E^Q[X] = \frac{1}{(1+r)^T} \sum_{n=0}^{T} \binom{T}{n} q^n (1-q)^{T-n} \max \{0, s_0 u^n d^{T-n-k} \}
\]
where $k$ is the strike price.

But what is the option price at time $0 \leq t \leq T$?

For any $t \leq T - 1$, there are at most $t$ ups.

$$V_t(k) = \frac{1}{1 + r} E^Q[V_{t+1}|(# \text{ ups up to time } t) = k] = \frac{1}{1 + r}(q_u V_{t+1}(k + 1) + (1 - q_u)V_{t+1}(k))$$

So, we have proved the recursion relation:

$$V_T(k) = \Phi(S_0 u^k d^{T-k}), \quad \forall \ k = 0, 1, ..., T$$

$\forall \ t \leq T - 1, \forall \ k \leq t$:

$$V_t(k) = \frac{1}{1 + R}(q_u V_{t+1}(k + 1) + (1 - q_u)V_{t+1}(k))$$

The only question we might need to answer is what are the hedging strategies.

For hedging we need to start at $t = 0$ and work forward by using the 1-period model hedging strategy to obtain $H = (x, y)$ which hedges the option given by:

$$\begin{cases} V_1(1), & \text{with probability } q_u \\ V_1(0), & \text{with probability } q_d \end{cases}$$

Specifically,

$$x = \frac{1}{1 + R} (uV_1(0) - dV_1(1)) \frac{1}{u - d}$$

$$y = \frac{1}{S_0} (V_1(1) - V_1(0)) \frac{1}{u - d}$$

This is obtain by solving the system:

$$(1 + r)x + S_0 uy = V_1(1)$$

$$(1 + r)x + S_0 dy = V_1(0)$$

In general: at time $t \leq T - 1$, for a fixed $k \in \{0, 1, ..., t\}$

$$x = \frac{1}{1 + r} (uV_{t+1}(k) - dV_{t+1}(k + 1)) \frac{1}{u - d}$$

$$y = \frac{1}{S_0} (V_{t+1}(k + 1) - V_{t+1}(k)) \frac{1}{u - d}$$

**Call options on a stock index**
Options, so far, were defined in term of one underlying security. But in general one can define options one two or more underlying securities. For example, given a function $g : \mathbb{R}^N \rightarrow \mathbb{R}^+$ one can take $X$ to be $X = g(S_1(T), ..., S_N(T))$. Then if you know the joint probability distribution of the random variables $S_1(T), ..., S_N(T)$ under the martingale measure, it is easy to compute the time 0 value of this contingent claim. In particular, with:

$$g(S_1, ..., S_N) = (a_1S_1 + ... + a_NS_N - e)^+$$

for positive scalars $a_1, ..., a_N$, you could have a call option on a stock index.

Or with

$$g(S_1, ..., S_N) = \max\{S_1, ..., S_N, e\}$$

you could have a contingent claim delivering the best of $N$ securities and the cash amount $e$.

**Example**

Suppose $K = 9$, $N = 2$, $T = 2$, $r = 0$ and the price process and information are as bellow. Also this model (one can show with a few computations) has an unique probability measure $Q$, computed as below.

Consider a call option with exercise price 13 on the time $T = 2$ value of the stock index $S_1(t) + S_2(t)$

What is the time 0 value of such a call option? What is the time 0 value of a contingent claim on the 2 stocks and 8?