MONTE CARLO SIMULATION

| TABLE 4.2 Results from Pricing a European Arithmetic Asian Option by Monte Carlo Simulation |
|-----------------------------------------------|-----------------|-----------------|-----------------|
| Price                     | Standard error | Relative         |                  |
|                            |                | computation     |                  |
| Simple Monte Carlo         | 5.038019       | 0.248236        | 1.00            |
| Antithetic                | 5.156263       | 0.135463        | 1.23            |
| Control variate           | 5.207977       | 0.010366        | 1.05            |
| Antithetic and control variate | 5.216232 | 0.006596        | 1.32            |

finally both an antithetic control variate and a geometric Asian control variate. The addition of the antithetic and geometric Asian control variate increase the computation time by approximately 30 per cent, but reduces the standard error by approximately 37 times. To achieve this reduction with the simple Monte Carlo would require increasing the number of simulations by 37 × 37 = 1369 times with a roughly equivalent increase in the computation time.

4.10 A LOOKBACK CALL OPTION UNDER STOCHASTIC VOLATILITY WITH DELTA, GAMMA AND VEGA CONTROL VARIATES

In this example we price a European fixed strike lookback call option. This option pays the difference, if positive, between the maximum of a set of observations (fixings) of the asset price $S_i$ at dates $i = 1, \ldots, N$ and the strike price. Thus the pay-off at the maturity date is

$$
\max(0, \max(S_i; i = 1, \ldots, N) - K)
$$

We will also assume that the asset price and the variance of the asset price returns $V = \sigma^2$ are governed by the following stochastic differential equations:

$$
dS = rS dt + \sigma S dz_1
$$

$$
dV = \sigma(V - V) dt + \xi \sqrt{V} dz_2
$$

and that the Wiener processes $dz_1$ and $dz_2$ are uncorrelated, but this is easily generalised as we saw in section 4.7. Figure 4.26 illustrates two typical asset price paths and the fixing dates.

There is no analytical solution for the price of European fixed strike lookback call option with discrete fixings and stochastic volatility. However, there is a simple analytical formula for the price of a continuous fixing strike lookback call with constant volatility:

$$
C_{\text{fixed strike lookback call}} = G + Se^{-\gamma T} N(x + \sigma \sqrt{T}) - Ke^{-\gamma T} N(x)
$$

$$
- \frac{S}{B} \left( e^{-\gamma T} \left( \frac{E}{S} \right)^B \right) N \left( x + (1 - B) \sigma \sqrt{T} \right)
$$

$$
- e^{-\gamma T} N \left( x + \sigma \sqrt{T} \right)
$$

Finally both an antithetic control variate and a geometric Asian control variate. The addition of the antithetic and geometric Asian control variate increase the computation time by approximately 30 per cent, but reduces the standard error by approximately 37 times. To achieve this reduction with the simple Monte Carlo would require increasing the number of simulations by $37 \times 37 = 1369$ times with a roughly equivalent increase in the computation time.
where

\[
E = K, \quad G = 0 \\
E = M, \quad G = e^{-\delta T} (M - K)
\]

\[
B = \frac{2(r - \delta)}{\sigma^2}, \quad x = \frac{\ln \left( \frac{S}{E} \right) + (r - \delta - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}}
\]

and \(M\) is the current known maximum. We can therefore use the continuously fixed floating strike lookback call option formula to compute \(\delta, \gamma\) and \(\nu\) hedge control variates. Rather than differentiate equation (4.52) with respect to the asset price twice and volatility once which would lead to extremely complex expressions it is more efficient to use finite difference approximations to the partial differentials for \(\gamma\) and \(\nu\).

Figure 4.27 shows a pseudo-code implementation of the Monte Carlo valuation of a European fixed strike lookback call option with continuously fixed lookback call option \(\delta, \gamma\) and \(\nu\) hedge control variates.\(^6\)

As discussed in section 4.4, the values of \(\beta_1, \beta_2\) and \(\beta_3\) were obtained by linear regression. Table 4.3 gives typical standard errors and relative computation times for the example of the application of the Monte Carlo valuation of a European fixed strike lookback call option in a Black–Scholes world with stochastic volatility using antithetic and \(\delta, \gamma\) and \(\nu\) control variates.

With the combined antithetic, \(\delta, \gamma\) and \(\nu\) control variates the standard error is reduced by a factor of 12. To achieve this reduction with the simple Monte Carlo method would require 144,000 simulations with an execution time of roughly 3.5 hours. With careful choice of the most efficient control variates and optimisation of the code the execution time can typically be reduced by a factor of between two and five.
can therefore use the continuously fixed option delta, gamma and vega hedge


tion of the Monte Carlo valuation of a continuously fixed lookback call option.

\[ \beta_1, \beta_2 \text{ and } \beta_3 \text{ were obtained by linear interpolation and relative computation times for Monte Carlo valuation of a European fixed strike option with stochastic volatility using antithetic variates.} \]

\[ \alpha \text{ and vega control variates the standard} \]

\[ \text{the Monte Carlo simulation with the simple Monte Carlo h an execution time of roughly 3.5 hours.} \]

\[ \text{optimisation of the code to the order of between two and five.} \]
\[ cv2 = cv1 + \gamma_1 \cdot (S_{ltm} - S_t)^2 + 2 \cdot S_t \cdot (\gamma_1 \cdot \exp(V_t \cdot dt) + \gamma_2) + \gamma_2 \cdot (S_{ltm} - S_t)^2 \cdot \exp(V_t \cdot dt) \]

\[ cv3 = cv2 + \gamma_2 \cdot (V_{tn} - V_t) + (\gamma_1 \cdot evg1 + evg2 - V_t) + \gamma_2 \cdot (V_{tn} - V_t) + (\gamma_1 \cdot evg1 + evg2 - V_t) \]

\[ V_t = V_{tn} \]
\[ S_t = S_{tn} \]
\[ S_t = S_{tn} \]

if (S_t > maxS1) maxS1 = S_t
if (S_t > maxS2) maxS2 = S_t

next i

\[ CT = 0.5 \cdot (\max(0, \maxS1 - K) + \max(0, \maxS2 - K) + \beta_1 \cdot cv1 + \beta_2 \cdot cv2 + \beta_3 \cdot cv3) \]
\[ \text{sum}_1 \cdot CT + CT \]
\[ \text{sum}_2 \cdot CT + CT \]

next j

\[ \text{call value} = \frac{\text{sum}_1 - \text{sum}_2 \cdot \exp(-r \cdot T)}{\sqrt{\text{sum}_2 \cdot \text{sum}_3}} \cdot \exp(-2 \cdot r \cdot T) \]
\[ SD = \frac{\sqrt{\text{sum}_2 - \text{sum}_1 \cdot \text{sum}_3 \cdot \text{sum}_1 \cdot \text{sum}_2}}{\text{sum}_2} \cdot \frac{\exp(-2 \cdot r \cdot T)}{\sqrt{\text{sum}_1 \cdot \text{sum}_2 \cdot \text{sum}_3}} \]
\[ SE = \frac{SD}{\sqrt{M}} \]

### TABLE 4.3 Typical Standard Errors and Computation Times
for the Monte Carlo Valuation of a European Fixed Strike Lookback Call Option in a Black–Scholes World with Stochastic Volatility Using Antithetic and Delta-, Gamma- and Vega-based Control Variates

<table>
<thead>
<tr>
<th>Strike price</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time to maturity</td>
<td>1 year</td>
</tr>
<tr>
<td>Initial asset price</td>
<td>100</td>
</tr>
<tr>
<td>Volatility</td>
<td>20%</td>
</tr>
<tr>
<td>Riskless interest rate</td>
<td>6%</td>
</tr>
<tr>
<td>Continuous dividend yield</td>
<td>3%</td>
</tr>
<tr>
<td>Mean reversion rate ((\alpha))</td>
<td>5.0</td>
</tr>
<tr>
<td>Volatility of volatility ((\beta))</td>
<td>0.02</td>
</tr>
<tr>
<td>Number of time steps</td>
<td>52</td>
</tr>
<tr>
<td>Number of simulations</td>
<td>1000</td>
</tr>
<tr>
<td>Continuous fixed strike lookback call value</td>
<td>17,729</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Standard error</th>
<th>Relative computation time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple estimate</td>
<td>0.4803</td>
</tr>
<tr>
<td>With antithetic variate</td>
<td>0.2030</td>
</tr>
<tr>
<td>With control variates</td>
<td>0.0485</td>
</tr>
<tr>
<td>Combined variates</td>
<td>0.0378</td>
</tr>
</tbody>
</table>