initialise parameters \( [K, T, E, \sigma, r, \text{div}, N, N_j, dx] \)

\[
\begin{align*}
\text{dt} &= T/N \\
\mu &= r - \text{div} - 0.5 \cdot \sigma^2 \\
\text{dx} &= \exp(dx) \\
\text{pu} &= 0.5 \cdot \text{dt} \cdot (\sigma/\text{dx})^2 + \mu/\text{dx} \\
\text{pm} &= 1.0 - \text{dt} \cdot (\sigma/\text{dx})^2 - r \cdot \text{dt} \\
\text{pd} &= 0.5 \cdot \text{dt} \cdot (\sigma/\text{dx})^2 - \mu/\text{dx} \\
\end{align*}
\]

\{ initialise asset prices at maturity \}

\[
\text{St}[-N_j] = S \cdot \exp(-N_j \cdot \text{dx})
\]

for \( j = -N_j + 1 \) to \( N_j \) do \( \text{St}[j] = \text{St}[j-1] \cdot \text{dx} \)

\{ initialise option values at maturity \}

for \( j = -N_j \) to \( N_j \) do \( C[0,j] = \max(0, K - \text{St}[j]) \)

\{ step back through lattice \}

for \( i = N-1 \) downto \( 0 \) do

for \( j = -N_j + 1 \) to \( N_j - 1 \) do

\[
C[1,j] = \text{pu} \cdot C[0,j+1] + \text{pm} \cdot C[0,j] + \text{pd} \cdot C[0,j-1]
\]

\{ boundary conditions \}

\[
\begin{align*}
C[1,-N_j] &= C[1,-N_j+1] + (\text{St}[-N_j+1] - \text{St}[-N_j]) \\
C[1,N_j] &= C[1,N_j-1] \\
\end{align*}
\]

\{ apply early exercise condition \}

for \( j = -N_j \) to \( N_j \) do

\[
C[0,j] = \max(C[1,j], K - \text{St}[j])
\]

next \( i \)

American put = \( C[0,0] \)

The pseudo-code implementation of the explicit finite difference method using equation (3.18) for the pricing of an American put option is given in Figure 3.9 (the changes from the European call pseudo-code in Figure 3.7 are highlighted in bold). The first highlighted change is the maturity condition for a put option. Secondly, we have introduced a storage efficiency improvement. The option value array \( C[,] \) only has two time indices, zero and one, rather than zero up to \( N \). Index one is used to temporarily
store the discounted expectation and then the boundary conditions and early exercise test store the appropriate value in time index zero.

**Example: Pricing an American Put Option in a Trinomial Tree**

We price a one-year maturity, at-the-money American put option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum, the asset pays a continuous dividend yield of 3 per cent per annum, the trinomial tree has three time steps and the space step is 0.2; \( K = 100, \ T = 1 \) year, \( S = 100, \ \sigma = 0.2, \ r = 0.06, \ delta = 0.03, \ N = 3, \ N_j = 3, \ \Delta x = 0.2 \). Figure 3.10 illustrates the numerical results.

Firstly the constants; \( \Delta t = \frac{T}{N} = \frac{1}{3} = 0.3333 \)

### FIGURE 3.10 Numerical Example for an American Put Option by Explicit Finite Difference Method

<table>
<thead>
<tr>
<th>( K )</th>
<th>( T )</th>
<th>( S )</th>
<th>( \sigma )</th>
<th>( r )</th>
<th>( \text{div} )</th>
<th>( N )</th>
<th>( N_j )</th>
<th>( dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1</td>
<td>100</td>
<td>0.2</td>
<td>0.06</td>
<td>0.03</td>
<td>3</td>
<td>3</td>
<td>0.2</td>
</tr>
</tbody>
</table>

\( \Delta t = \frac{T}{N} = \frac{1}{3} = 0.3333 \)

<table>
<thead>
<tr>
<th>( dt )</th>
<th>( nu )</th>
<th>( edx )</th>
<th>( pu )</th>
<th>( pm )</th>
<th>( pd )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3333</td>
<td>0.1000</td>
<td>1.2214</td>
<td>0.1750</td>
<td>0.6467</td>
<td>0.1583</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( j )</th>
<th>( St, t )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i )</td>
<td>( 0 )</td>
<td>( 0.0720 )</td>
<td>( 0.0000 )</td>
<td>( 0.0000 )</td>
<td>( 0.0000 )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( 182.21 )</td>
<td>( 0.0720 )</td>
<td>( 0.0000 )</td>
<td>( 0.0000 )</td>
<td>( 0.0000 )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( 149.18 )</td>
<td>( 0.0720 )</td>
<td>( 0.0000 )</td>
<td>( 0.0000 )</td>
<td>( 0.0000 )</td>
</tr>
<tr>
<td>( 3 )</td>
<td>( 122.14 )</td>
<td>( 1.0422 )</td>
<td>( 0.4544 )</td>
<td>( 0.0000 )</td>
<td>( 0.0000 )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( 100.00 )</td>
<td>( 5.0058 )</td>
<td>( 4.7261 )</td>
<td>( 2.8701 )</td>
<td>( 0.0000 )</td>
</tr>
<tr>
<td>( -1 )</td>
<td>( 81.87 )</td>
<td>( 17.7691 )</td>
<td>( 17.4443 )</td>
<td>( 18.1269 )</td>
<td>( 18.1269 )</td>
</tr>
<tr>
<td>( -2 )</td>
<td>( 67.03 )</td>
<td>( 31.6353 )</td>
<td>( 31.6353 )</td>
<td>( 31.6353 )</td>
<td>( 31.9680 )</td>
</tr>
<tr>
<td>( -3 )</td>
<td>( 54.88 )</td>
<td>( 43.7862 )</td>
<td>( 43.7862 )</td>
<td>( 43.7862 )</td>
<td>( 45.1188 )</td>
</tr>
</tbody>
</table>

*discounted expectation or boundary condition*
\[ \nu = r - \delta - \frac{1}{2}\sigma^2 = 0.06 - 0.03 - 0.5 \times 0.02^2 = 0.01 \]
\[ edx = \exp(\Delta x) = \exp(0.2) = 1.2214 \]
\[ p_u = \frac{1}{2} \Delta t \left( \left( \frac{\sigma}{\Delta x} \right)^2 + \nu \frac{\nu}{\Delta x} \right) = \frac{1}{2} \times 0.3333 \times \left( \frac{0.2}{0.2} \right)^2 + \frac{0.01}{0.2} = 0.1750 \]
\[ p_m = 1 - \Delta t \left( \frac{\sigma}{\Delta x} \right)^2 - r\Delta t = 1 - 0.3333 \times \left( \frac{0.2}{0.2} \right)^2 - 0.06 \times 0.3333 = 0.6467 \]
\[ p_d = \frac{1}{2} \Delta t \left( \frac{\sigma}{\Delta x} \right)^2 - \nu \frac{\nu}{\Delta x} = \frac{1}{2} \times 0.3333 \times \left( \frac{0.2}{0.2} \right)^2 - \frac{0.01}{0.2} = 0.1583 \]

Then the asset prices at maturity are computed (these apply to every time step). The asset price at node (3, -3) is computed as:
\[ S_{3,-3} = S \times \exp(-N \times \Delta x) = 100 \times \exp(-3 \times 0.2) = 54.88 \]

then the other asset prices are computed from this. For example at node (3, -2) the asset price is given by
\[ S_{3,-2} = S_{3,-2} \times edx = 54.8812 \times 1.2214 = 67.03 \]

Next the option values at maturity are computed. For node (3, -2) we have:
\[ C_{3,-2} = \max(0, K - S_{3,-2}) = \max(0, 100 - 67.03) = 32.9680 \]

Finally we perform discounted expectations back through the tree. For node (2, -1) we have
\[ C_{2,-1} = p_u \times C_{3,0} + p_m \times C_{3,-1} + p_d \times C_{3,-2} \]
\[ = 0.1750 \times 0.0000 + 0.6467 \times 18.1269 + 0.1583 \times 32.9680 = 16.9420 \]

Applying the early exercise test we have
\[ C_{2,-1} = \max(C_{2,-1}, K - S_{2,-1}) = \max(16.9420, 100 - 81.8731) = 18.1269 \]

The accuracy of this method is \( O(\Delta x^2 + \Delta t) \) which means that if we halve \( \Delta x^2 + \Delta t \) we halve the error. Therefore to halve the error we must halve the time step, but only need to reduce the space step by a factor of \( 1/\sqrt{2} \).

### 3.5 STABILITY AND CONVERGENCE

It is very important to ensure that the probabilities \( p_u, p_m \) and \( p_d \) are positive and that the stability and convergence condition mentioned in section 3.2 is satisfied:
\[
\Delta x \geq \sigma \sqrt{3 \Delta t} \quad (3.27)
\]

Imagine that \( C \) represents the exact solution of the PDE and \( O \) represents the exact solution of the finite difference equation. The difference \( C - O \) is called the discretisation