Then for each time step 1 to \( N \), where \( N = 10 \), \( \ln(S_t) \) is simulated. For example for \( j = 1 \) and \( i = 1 \) (dropping the \( i \) and \( j \) subscripts):

\[
\ln(S_t) = \ln(S_0) + \mu dt + \sigma dW_t \\
\ln(S_t) = 4.6052 + 0.001 + 0.0632 \times (-0.0497) = 4.6030
\]

At \( i = 10 \)

\[
S_T = \exp(\ln(S_t)) = \exp(4.6521) = 104.81 \\
C_T = \max(0, S_T - K) = \max(0, 104.81 - 100) = 4.8070
\]

The sum of the values of \( C_T \) and the squares of the values of \( C_T \) are accumulated:

\[
\sum_{j=1}^{M} C_{T,j} = 996.488 \quad \text{(sum.CT)} \quad \text{and} \quad \sum_{j=1}^{M} (C_{T,j})^2 = 26610.7 \quad \text{(sum.CT2)}
\]

The estimate of the option value \( \hat{C}_0 \) (call.value) is then given by

\[
\hat{C}_0 = 996.488/100 \times \exp(-0.06 \times 1) = 9.3846
\]

The standard deviation (SD) is given by

\[
SD = \frac{\sqrt{\sum_{j=1}^{M} (C_{T,j})^2 - \frac{1}{M} \left( \sum_{j=1}^{M} C_{T,j} \right)^2 \exp(-2rT)}}{\sqrt{M-1}}
\]

\[
= \sqrt{26610.73 - \frac{1}{100} (996.488)^2 \exp(-2 \times 0.06 \times 1)} \quad \frac{1}{100-1} = 12.2246
\]

and so the standard error (SE) is

\[
SE = \frac{SD}{\sqrt{M}} = 12.2246/10 = 1.22246
\]

Unfortunately, in order to get an acceptably accurate estimate of the option price a very large number of simulations has to be performed, typically in the order of millions \( (M > 1000000) \). This problem can be dealt with by using variance reduction methods. These methods work on exactly the same principle as that of hedging an option position, that is that the pay-off of a hedged portfolio will have a much smaller variability than an unhedged pay-off. This corresponds to the variance (or equivalently standard error) of a simulated hedge portfolio being much smaller than that of the unhedged pay-off. We will stress this interpretation throughout this chapter.

### 4.3 Antithetic Variates and Variance Reduction

Imagine that you have written an option on an asset \( S_1 \) and simultaneously are able to write an option on an asset \( S_2 \) which is perfectly negatively correlated with \( S_1 \), and
which currently has exactly the same price as $S_1$. That is $S_1$ and $S_2$ satisfy the stochastic differential equations

$$dS_{1,t} = rS_{1,t} dt + \sigma S_{1,t} dz_t$$
$$dS_{2,t} = rS_{2,t} dt - \sigma S_{2,t} dz_t$$

The value of these two options are identical since the price and volatility of the two assets are identical. However, the variance of the pay-off of a portfolio consisting of the two options is much less than the variance of the pay-off of each individual option since, roughly speaking, when one option pays off the other does not and vice versa. It may not be obvious at first why this leads to a smaller variance and so we give some intuition. Figure 4.4 illustrates the pay-off of a written call option on a lognormally distributed asset and the probability distribution of the payoff.

The variance (or variability) of the pay-off is very high because of the large spike of probability which corresponds to all the asset prices below the strike price. The hedge portfolio we have just described removes this spike and so reduces the variance of the pay-off.

This technique of creating a hypothetical asset which is perfectly negatively correlated with the original asset is called antithetic variance reduction and the created asset is called an antithetic variate. Implementation of this technique is very simple, for example, consider pricing a European call option. Our simulated pay-offs are

$$C_{T,j} = \max(0, S \exp(\nu T + \sigma \sqrt{T} \epsilon_j) - K)$$

We can simulate the pay-offs to the option on the perfectly negatively correlated asset as

$$\bar{C}_{T,j} = \max(0, S \exp(\nu T + \sigma \sqrt{T} (-\epsilon_j)) - K)$$

In other words we simply replace $\epsilon_j$ by $-\epsilon_j$ in the equation for the simulation of the asset. We then take the average of the two pay-offs as the pay-off for that simulation. Note that, not only do we obtain a much more accurate estimate from $M$ pairs of $(C_{T,j}, \bar{C}_{T,j})$ than from $2M$ of $C_{T,j}$, but it is also computationally cheaper to generate the pair $(C_{T,j}, \bar{C}_{T,j})$ than two instances of $C_{T,j}$. This method also ensures that the mean of the normally

---

**FIGURE 4.4** Pay-off of Written Call Option and Probability Distribution of the Pay-off

![Diagram showing pay-off and probability distribution](image-url)
and \( \sigma_2 \) satisfy the stochastic
\[
(4.11) \\
(4.12)
\]
ce and volatility of the two assets when doing this, and the created asset is correctly negatively correlated with the asset price. The hedge reduces the variance of the

\[
(4.13)
\]

effectively negatively correlated asset as

\[
(4.14)
\]
n for the simulation of the off for that simulation. Note on \( M \) pairs of \((C_T,j,C_T,j)\) to generate the pair \((C_T,j)\), at the mean of the normally

\[
\text{distribution of the Pay-off} \\
\text{Probability} \\
\text{Pay-off}
\]
distributed samples \( \varepsilon \) is exactly zero which also helps to improve the simulation. Figure 4.5 gives a pseudo-code implementation of the Monte Carlo valuation of a European call option with antithetic variance reduction. The differences from Figure 4.2 are highlighted in bold.

**Example: Pricing a European Call Option by Monte Carlo Simulation with Antithetic Variance Reduction**

We price a one-year maturity, at-the-money European call option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum, the asset pays a continuous dividend yield of 3 per cent per annum. The simulation has one time step and 100 simulations; \( K = 100, T = 1 \) year, \( S = 100, \sigma = 0.2, r = 0.06, \delta = 0.03, N = 1, M = 100 \). Figure 4.6 illustrates...
<table>
<thead>
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<th>K</th>
<th>T</th>
<th>S</th>
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<th>r</th>
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</tr>
</tbody>
</table>
the numerical results, the simulated asset prices are only shown for \( j = 1, \ldots, 5 \) and \( j = 95, \ldots, 100 \). Note that in this example there is only one time step \((N = 1)\) because we only need to simulate asset prices at the maturity date of the option.

Firstly, the constants: \( \Delta t (dt) \), \( \sqrt{\Delta t} \) \((T)\), \( \sigma \sqrt{\Delta t} \) \((T)\) \((\text{m})\), \( \Delta t \) \((\ln S)\), and \( \ln(S)\) \((\ln S)\) are precomputed:

\[
\Delta t = \frac{T}{N} = \frac{1}{1} = 1
\]

\[
nudt = (r - \delta - \frac{1}{2} \sigma^2) \Delta t = (0.06 - 0.03 - 0.5 \times 0.2^2) \times 1 = 0.01
\]

\[
sigsdt = \sigma \sqrt{\Delta t} = 0.2 \sqrt{1} = 0.2
\]

\[
\ln S = \ln(S) = 4.6052
\]

Then for each simulation \( j = 1 \) to \( M \), where \( M = 100 \), \( \ln(S_{1,t}) \) and \( \ln(S_{2,t}) \) are initialised to \( \ln(S) = 4.6052 \). Then \( \ln(S_{1,t}) \) and \( \ln(S_{2,t}) \) are simulated, for example for \( j = 1 \) and \( i = 1 \):

\[
\ln(S_{1,t}) = 4.6052 + 0.010 + 0.2 \times (-0.8265) = 4.4499
\]

\[
\ln(S_{2,t}) = 4.6052 + 0.010 + 0.2 \times (0.8265) = 4.7805
\]

\[
S_{1,t} = \exp(4.4499) = 85.62
\]

\[
S_{2,t} = \exp(4.7805) = 119.16
\]

Computing the pay-off at maturity gives:

\[
C_T = 0.5 \times (\max(0, 85.62 - 100) + \max(0, 119.16 - 100)) = 9.5807
\]

The sum of the values of \( C_T \) and the squares of the values of \( C_T \) are accumulated:

\[
\sum_{j=1}^{M} C_{T,j} = 1140.37 \quad \text{and} \quad \sum_{j=1}^{M} (C_{T,j})^2 = 20790.8
\]

The estimate of the option value \( \hat{C}_0 \) \((\text{call valeur})\) is then given by

\[
\hat{C}_0 = \frac{1140.366}{100} \times \exp(-0.06 \times 1) = 10.7396
\]

This technique can be easily applied to virtually any Monte Carlo simulation to improve the efficiency. In the next section we describe more advanced variance reduction methods based on the hedging analogy.

### 4.4 CONTROL VARIATES AND HEDGING

The general approach of using hedges as control variates was first described by Clewlow and Carverhill (1994). Consider the case of writing a European call option. Figure 4.4