order of variance reduction in the simple Monte Carlo method would require increasing the number of simulations by a factor of 8100, that is, 8.1 million simulations with a computation time of approximately 3.15 hours. However, this is a slightly unrealistic example because we have the delta and gamma analytically, and so the hedge works perfectly in the limit as the time step is decreased to zero. In following sections we describe more realistic examples.

4.6 COMPUTING HEDGE SENSITIVITIES

The standard hedge sensitivities, delta, gamma, vega, theta and rho can be computed by approximating them by finite difference ratios:

\[
\text{delta} = \frac{\partial C}{\partial S} \approx \frac{C(S + \Delta S) - C(S - \Delta S)}{2\Delta S}
\]

\[
\text{gamma} = \frac{\partial^2 C}{\partial S^2} \approx \frac{C(S + \Delta S) - 2C(S) + C(S - \Delta S)}{\Delta S^2}
\]

\[
\text{vega} = \frac{\partial C}{\partial \sigma} \approx \frac{C(\sigma + \Delta \sigma) - C(\sigma - \Delta \sigma)}{2\Delta \sigma}
\]

\[
\text{theta} = \frac{\partial C}{\partial t} \approx \frac{C(t + \Delta t) - C(t)}{\Delta t}
\]

\[
\text{rho} = \frac{\partial C}{\partial r} \approx \frac{C(r + \Delta r) - C(r - \Delta r)}{2\Delta r}
\]

where \(C(S + \Delta S)\) is the Monte Carlo estimate using an initial asset price of \(S + \Delta S\), and \(\Delta S\) is a small fraction of \(S\), e.g., \(\Delta S = 0.001S\) and the other \(C(.)\)'s are defined similarly. Note that every price \(C(.)\) in equations (4.27)-(4.31) should be computed using the same set of random numbers. If this is not done then the random error in the prices from the Monte Carlo simulation can be a large proportion of the price differences in the numerator of the finite difference ratios leading to very large errors in the sensitivity estimates. By using the same random numbers the pricing errors will tend to cancel out.

A more efficient way to compute delta and from this gamma is by applying the discounted expectations approach. We can express the standard European call delta as follows:

\[
\text{delta} = \frac{\partial C}{\partial S} = \frac{\partial}{\partial S} \left( e^{-rT} E \left[ (S_T - K)1_{S_T > K} \right] \right)
\]

where \(S_T = S \exp(\nu T + \sigma \varepsilon_T)\) and \(1_{S_T > K}\) is the indicator function which is one if \(S_T > K\) and zero otherwise. Substituting \(S_T\) in equation (4.32) and differentiating we obtain:

\[
\text{delta} = e^{-rT} E[\exp(\nu T + \sigma \varepsilon_T)1_{S_T > K}]
\]

So to compute delta by Monte Carlo simulation we simulate the asset price as usual and compute the discounted expectation of an instrument which pays off \(\exp(\nu T + \sigma \varepsilon_T)\) if \(S_T > K\) and zero otherwise. Figure 4.15 gives a pseudo-code implementation of this method.
FIGURE 4.15  Pseudo-code for Monte Carlo Calculation of a European Call Option Delta: a Black–Scholes World

```plaintext
initialise parameters { K, T, S, sig, r, div, M }

{ precompute constants }

\[ \Delta t = T \]
\[ \nu dt = (r - \text{div} - 0.5 \times \text{sig}^2) \Delta t \]
\[ \text{sig} \Delta t = \text{sig} \times \sqrt{\Delta t} \]

\[ \text{sum} \_\text{CT} = 0 \]
\[ \text{sum} \_\text{CT}^2 = 0 \]

for \( j = 1 \) to \( M \) do { for each simulation }

\[ \varepsilon = \text{standard normal sample} \]
\[ e = \exp(\nu dt + \text{sig} \Delta t \varepsilon) \]
\[ ST = S \times e \]

if \( |ST - K| < K \) then
\[ \text{CT} = e \]
else
\[ \text{CT} = 0 \]

\[ \text{sum} \_\text{CT} = \text{sum} \_\text{CT} + \text{CT} \]
\[ \text{sum} \_\text{CT}^2 = \text{sum} \_\text{CT}^2 + \text{CT} \times \text{CT} \]

next \( j \)

\[ \text{delta value} = \frac{\text{sum} \_\text{CT}}{M \times \exp(-r \times T)} \]
\[ \text{SD} = \sqrt{(\text{sum} \_\text{CT}^2 - \text{sum} \_\text{CT} \times \text{sum} \_\text{CT}/M) \times \exp(-2 \times r \times T)/(M-1)} \]
\[ \text{SE} = \frac{\text{SD}}{\sqrt{M}} \]
```

Example: Computing a European Call Option Delta by Monte Carlo Simulation

We compute the delta of a one year maturity, at-the-money European call option with the current asset price at 100 and volatility of 20 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum, the asset pays a continuous dividend yield of 3 per cent per annum. The simulation has one time step and 100 simulations; \( K = 100 \), \( T = 1 \) year, \( S = 100 \), \( \sigma = 0.2 \), \( r = 0.06 \), \( \delta = 0.03 \), \( N = 1 \), \( M = 100 \). Figure 4.16 illustrates the numerical results for the simulation of paths \( j = 1, \ldots, 5 \) and 95, \ldots, 100.

Firstly, the constants; \( \Delta t (dt) \), \( \nu \Delta t (\nu dt) \), \( \sigma \sqrt{\Delta t} (\text{sig} \Delta t) \) are precomputed:

\[ \Delta t = \frac{T}{N} = \frac{1}{1} = 1 \]
\[ \nu dt = (r - \delta - \frac{1}{2} \sigma^2) \Delta t = (0.06 - 0.03 - 0.5 \times 0.2^2) \times 1 = 0.01 \]
\[ \text{sig} \Delta t = \sigma \sqrt{\Delta t} = 0.2 \sqrt{1} = 0.2 \]
FIGURE 4.16 Monte Carlo Calculation of a European Call Option Delta in a Black–Scholes World

<table>
<thead>
<tr>
<th>K</th>
<th>T</th>
<th>S</th>
<th>sig</th>
<th>r</th>
<th>div</th>
<th>N</th>
<th>M</th>
<th>sum_CT</th>
<th>sum_CT2</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1</td>
<td>100</td>
<td>0.2</td>
<td>0.06</td>
<td>0.03</td>
<td>1</td>
<td>100</td>
<td>60.3992</td>
<td>77.98</td>
<td>0.6097</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>dt</th>
<th>nudt</th>
<th>sigsdt</th>
<th>delta_value</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0100</td>
<td>0.2000</td>
<td>0.5688</td>
<td>0.0610</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>j</th>
<th>e</th>
<th>e</th>
<th>ST</th>
<th>CT</th>
<th>CT*CT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.8265</td>
<td>0.8562</td>
<td>85.6152</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>-0.6445</td>
<td>0.8879</td>
<td>88.7892</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>-0.9527</td>
<td>0.8348</td>
<td>83.4816</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
<td>-1.8013</td>
<td>0.7045</td>
<td>70.4504</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>5</td>
<td>2.4056</td>
<td>1.6341</td>
<td>163.4134</td>
<td>1.6341</td>
<td>2.6704</td>
</tr>
</tbody>
</table>

| 95 | 2.3200 | 1.6064 | 160.6420 | 1.6064 | 2.5806 |
| 96 | 1.9226 | 1.4837 | 148.3680 | 1.4837 | 2.2013 |
| 97 | -0.6575 | 0.8856 | 88.5599 | 0.0000 | 0.0000 |
| 98 | -1.0324 | 0.8216 | 82.1617 | 0.0000 | 0.0000 |
| 99 | -0.3316 | 0.9452 | 94.5232 | 0.0000 | 0.0000 |
| 100| -0.4677 | 0.9199 | 91.9860 | 0.0000 | 0.0000 |
then for each simulation $j = 1$ to $M$, where $M = 100$, the exponential term $\exp(u \delta t_1)$ is simulated, $S_T$ and $C_T$ are computed, and the sums $\text{sum}_{\text{CT}}$ and $\text{sum}_{\text{CT2}}$ are accumulated.

For $j = 1$ we have

$$e = \exp(nu \delta t + s_{\text{g}} \delta t \times e) = \exp(0.010 + 0.2 \times (-0.8265)) = 0.8562$$

$$S_T = S \times e = 100 \times 0.8562 = 85.62$$

$$S_T < K \text{ therefore } C_T = 0.$$  

For $j = 5$ we have

$$e = \exp(nu \delta t + s_{\text{g}} \delta t \times e) = \exp(0.01 + 0.2 \times (2.4056)) = 1.6341$$

$$S_T = S \times e = 100 \times 1.6341 = 163.41$$

$$S_T > K \text{ therefore } C_T = e = 1.6341.$$  

The sum of the values of $C_T$ and the squares of the values of $C_T$ are accumulated: $\text{sum}_{\text{CT}}$ and $\text{sum}_{\text{CT2}}$, giving $\text{sum}_{\text{CT}} = 60.399$ and $\text{sum}_{\text{CT2}} = 78.0$. The estimate of the delta value is then given by

$$\delta = \text{sum}_{\text{CT}}/M \times \exp(-r \times T) = 60.399/100 \times \exp(-0.06 \times 1) = 0.5688$$

The antithetic and control variate methods can be applied in the same way as for the Monte Carlo valuation of the option itself.

This technique cannot be used for the calculation of $\gamma$ because differentiating equation (4.33) again leads to the expectation of a Dirac delta function which cannot easily be evaluated by Monte Carlo simulation. We can, however, use a finite difference ratio in terms of $\delta$

$$\gamma = \frac{\partial^2 C}{\partial S^2} \approx \frac{\delta C}{\delta S} = \frac{\delta (S + \Delta S) - \delta (S - \Delta S)}{2 \Delta S}$$

(4.34)

### 4.7 MULTIPLE STOCHASTIC FACTORS

One of the main uses of Monte Carlo simulation is for pricing options under multiple stochastic factors. For example, pricing options whose pay-off depends on multiple asset prices, or with stochastic volatility or interest rates. For example, consider a European spread option on the difference between two assets (e.g. stock indices) $S_1$ and $S_2$, which follow GBM:  

$$dS_1 = (r - \delta_1)S_1 \, dt + \sigma_1S_1 \, dZ_1$$

(4.35)

$$dS_2 = (r - \delta_2)S_2 \, dt + \sigma_2S_2 \, dZ_2$$

(4.36)

The first complication we have is that it is quite likely that we will want $S_1$ and $S_2$ to be correlated to some degree $\rho$. That is, the Brownian motions $dZ_1$ and $dZ_2$ have instantaneous correlation $\rho$ ($dZ_1, dZ_2 = \rho dt$). In order to price the option by simulation we use the solutions of the SDEs to simulate the asset prices, as in section 4.2.

$$S_{1,T} = S_1 \exp(u_1T + \sigma_1Z_{1,T})$$

(4.37)
where $v_1 = r - \delta_1 - \frac{1}{2} \sigma_1^2$ and

$$ S_{2,T} = S_2 \exp(v_2 T + \sigma_2 z_{2,T}) $$

(4.38)

where $v_2 = r - \delta_2 - \frac{1}{2} \sigma_2^2$. However, here we need to generate the variates $z_1$ and $z_2$ from a standard bivariate normal distribution with correlation $\rho$. This is easily achieved by generating independent standard normal variates $\varepsilon_1$ and $\varepsilon_2$ and combining them as follows:

$$ z_1 = \varepsilon_1 $$

(4.39)

$$ z_2 = \rho \varepsilon_1 + \sqrt{1 - \rho^2} \varepsilon_2 $$

(4.40)

The general procedure for generating $n$ correlated normal variates is described in section 4.11.

The Monte Carlo procedure is exactly the same as that for the standard European call in section 4.2 except that we simulate the two asset processes and from this the pay-off of the spread option ($\max(0, S_1 - S_2 - K)$). Figure 4.17 gives the pseudo-code implementation.

**Example: Pricing a European Spread Call Option by Monte Carlo Simulation**

We price a one-year maturity, European spread call option with a strike price of 1, current asset prices of 100, volatilities of 20 and 30 per cent, continuous dividend yields of 3 and 4 per cent and a correlation of 50 per cent. The continuously compounded interest rate is assumed to be 6 per cent per annum and the simulation has one time step and 100 simulations, i.e. $K = 1, T = 1, S_1 = 100, S_2 = 110, \sigma_1 = 0.20, \sigma_2 = 0.30, d_1 = 0.03, d_2 = 0.04, \rho = 0.50, r = 0.06, N = 1, M = 100$. Figure 4.18 illustrates the results of the calculations for the simulation of paths $j = 1, \ldots, 5$ and 95, $\ldots, 100$.

Firstly, the constants $\Delta t$ ($dt$), $\nu_1 \Delta t$ ($n_1 dt$), $\nu_2 \Delta t$ ($n_2 dt$), $\sigma_1 \sqrt{\Delta t}$ ($\sigma_1 \Delta t$), $\sigma_2 \sqrt{\Delta t}$ ($\sigma_2 \Delta t$), $\sqrt{1 - \rho^2}$ ($\rho_{ho}$) are precomputed:

$$ \Delta t = \frac{T}{N} = \frac{1}{1} = 1 $$

$$ n_1 dt = (r - \delta_1 - \frac{1}{2} \sigma_1^2) \Delta t = (0.06 - 0.03 - 0.5 \times 0.2^2) \times 1 = 0.01 $$

$$ n_2 dt = (r - \delta_2 - \frac{1}{2} \sigma_2^2) \Delta t = (0.06 - 0.04 - 0.5 \times 0.3^2) \times 1 = -0.025 $$

$$ \sigma_1 \Delta t = \sigma_1 \sqrt{\Delta t} = 0.2 \sqrt{1} = 0.2 $$

$$ \sigma_2 \Delta t = \sigma_2 \sqrt{\Delta t} = 0.3 \sqrt{1} = 0.3 $$

$$ \rho_{ho} = \sqrt{1 - \rho^2} = \sqrt{1 - 0.5^2} = 0.8660 $$

For each simulation $j = 1$ to $M (M = 100)$, $S_1$ and $S_2$ are simulated. For example, for $j = 1$ we have

$$ \varepsilon_1 = -0.8265, \varepsilon_2 = -0.0833 $$

$$ z_1 = \varepsilon_1 = -0.8265 $$
FIGURE 4.17 Pseudo-code for Monte Carlo Valuation of a European Spread Option in a Black–Scholes World

```plaintext
initialise parameters
    | K, T, S1, S2, sig1, sig2, div1, div2, rho, r, N, M |

{ precompute constants }

N = 1  { no path dependency }

dt = T/N
nu1dt = (r-div1-0.5*sig1^2)*dt
nu2dt = (r-div2-0.5*sig2^2)*dt
sig1stdt = sig1*sqrt(dt)
sig2stdt = sig2*sqrt(dt)
rho = sqrt( 1 - rho^2 )

sum_CT = 0
sum_CT2 = 0

for j = 1 to M do  { for each simulation }
    St1 = S1
    St2 = S2
    for i = 1 to N do  { for each time step }
        e1 = standard_normal_sample
        e2 = standard_normal_sample
        z1 = e1
        z2 = rho * e1 + srho * e2
        St1 = St1 * exp(nu1dt + sig1stdt*z1)
        St2 = St2 * exp(nu2dt + sig2stdt*z2)
    next i
    CT = max( 0, St1 - St2 - K )
    sum_CT = sum_CT + CT
    sum_CT2 = sum_CT2 + CT*CT
next j

call_value = sum_CT/M*exp(-r*T)
SD = sqrt( (sum_CT2 - sum_CT*sum_CT/M)*exp(-2*r*T)/(M-1) )

z2 = rho * e1 + srho * e2 = 0.5 * (-0.8265) + 0.8660 * (-0.0833) = -0.4854
S_{1,T} = S_1 * exp(nu1dt + sig1stdt * z1)
          = 100 * exp(0.0100 + 0.2 * (-0.8265)) = 85.615
S_{2,T} = S_2 * exp(nu2dt + sig2stdt * z2)
          = 110 * exp(-0.0250 + 0.3 * (-0.4854)) = 92.746
```

\[
X = \frac{\ln\left(\frac{S_1}{S_2}\right)}{-0.0833} = -0.4854
\]

**Figure 4.18 Monte Carlo Valuation of a European Spread Option in a Black–Scholes World**

<table>
<thead>
<tr>
<th>K</th>
<th>T</th>
<th>S1</th>
<th>S2</th>
<th>sig1</th>
<th>sig2</th>
<th>div1</th>
<th>div2</th>
<th>rho</th>
<th>r</th>
<th>N</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>100</td>
<td>110</td>
<td>0.2</td>
<td>0.3</td>
<td>0.03</td>
<td>0.04</td>
<td>0.50</td>
<td>0.06</td>
<td>1</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>dt</th>
<th>nu1dt</th>
<th>nu2dt</th>
<th>sig1stdt</th>
<th>sig2stdt</th>
<th>srho</th>
<th>sum_CT</th>
<th>sum_CT2</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000</td>
<td>0.0100</td>
<td>-0.0250</td>
<td>0.0200</td>
<td>0.3000</td>
<td>0.0650</td>
<td>742.97</td>
<td>21844.2</td>
<td>12.09319</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>j</th>
<th>e1</th>
<th>e2</th>
<th>z1</th>
<th>z2</th>
<th>S1</th>
<th>S2</th>
<th>CT</th>
<th>CT^2CT</th>
<th>call_value</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.8265</td>
<td>-0.0833</td>
<td>-0.8265</td>
<td>-0.4854</td>
<td>85.62</td>
<td>92.75</td>
<td>0.0000</td>
<td>0.0000</td>
<td>6.0670</td>
<td>1.2093</td>
</tr>
<tr>
<td>2</td>
<td>-0.5445</td>
<td>0.6059</td>
<td>-0.6445</td>
<td>0.3749</td>
<td>88.79</td>
<td>120.05</td>
<td>0.0000</td>
<td>0.0000</td>
<td>13.2711</td>
<td>176.12</td>
</tr>
<tr>
<td>3</td>
<td>-0.9527</td>
<td>1.3859</td>
<td>-0.9527</td>
<td>-1.6756</td>
<td>93.48</td>
<td>64.88</td>
<td>17.6036</td>
<td>309.89</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-1.8013</td>
<td>0.9632</td>
<td>-1.8013</td>
<td>-0.0665</td>
<td>70.45</td>
<td>105.16</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>5</td>
<td>2.4056</td>
<td>0.5148</td>
<td>2.4056</td>
<td>0.7569</td>
<td>163.41</td>
<td>134.63</td>
<td>27.7798</td>
<td>771.71</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>K</th>
<th>T</th>
<th>S1</th>
<th>S2</th>
<th>sig1</th>
<th>sig2</th>
<th>div1</th>
<th>div2</th>
<th>rho</th>
<th>r</th>
<th>N</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>2.3000</td>
<td>-0.4360</td>
<td>2.3000</td>
<td>0.7807</td>
<td>160.64</td>
<td>135.60</td>
<td>24.0443</td>
<td>578.13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>96</td>
<td>1.9026</td>
<td>-0.2514</td>
<td>1.9225</td>
<td>0.7436</td>
<td>148.37</td>
<td>134.10</td>
<td>13.2711</td>
<td>176.12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>97</td>
<td>-0.8575</td>
<td>-0.0533</td>
<td>-0.8575</td>
<td>-0.3748</td>
<td>88.56</td>
<td>95.87</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
</tr>
<tr>
<td>98</td>
<td>-1.0324</td>
<td>0.2024</td>
<td>-1.0324</td>
<td>-0.3409</td>
<td>82.16</td>
<td>96.85</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>-0.3316</td>
<td>0.1838</td>
<td>-0.3316</td>
<td>-0.0066</td>
<td>94.52</td>
<td>107.07</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
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<tr>
<td>100</td>
<td>-0.4677</td>
<td>0.5573</td>
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<td>0.2488</td>
<td>91.99</td>
<td>115.60</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
</tr>
</tbody>
</table>
CT = \max(0, S_{1, T} - S_{1, T} - K) = \max(0, 85.615 - 92.746 - 1) = 0.0

For j = 3 we have

\varepsilon_1 = -0.9527, \quad \varepsilon_2 = -1.3859

z_1 = \varepsilon_1 = -0.9527

z_2 = \rho \cdot \varepsilon_1 + \sigma_{ho} \cdot \varepsilon_2 = 0.5 \times (-0.9527) + 0.8660 \times (-1.3859) = -1.6766

S_{1, T} = S_1 \times \exp(nu_1 dt + \sigma_1 a dt \times z_1)

= 100 \times \exp(0.01000 + 0.2 \times (-0.9527)) = 83.482

S_{2, T} = S_2 \times \exp(nu_2 dt + \sigma_2 a dt \times z_2)

= 110 \times \exp(-0.0250 + 0.3 \times (-1.6766)) = 64.878

C_T = \max(0, S_{1, T} - S_{2, T} - K) = \max(0, 83.482 - 64.878 - 1) = 17.604

The sum of the values of CT and the squares of the values of CT are accumulated to obtain the call price, giving sum_CT = 742.968 and sum_CT2 = 21844.2. The estimate of the option value is then given by

call_value = \text{sum}_{CT} / M \times \exp(-r \times T) = 742.968 / 100 \times \exp(-0.06 \times 1) = 6.9976

In the same way if we want to price an option under more general stochastic processes such as stochastic volatility and/or stochastic interest rates we simply simulate the require processes. For example, imagine we want to price the European spread option when the underlying asset prices, S_1 and S_2, follow GBM, but where the variance of returns, \nu_1 and \nu_2, of the assets follow mean reverting square root processes (\alpha; Hull and White, 1988). The SDE’s for the asset prices and variances are given by equations (4.41)–(4.44) respectively:

\begin{align*}
    dS_1 &= rS_1 dt + \sigma_1 S_1 dz_1 \\
    dS_2 &= rS_2 dt + \sigma_2 S_2 dz_2 \\
    d\nu_1 &= \alpha_1 (\nu_1 - V_1) dt + \xi_1 \sqrt{\nu_1} dz_3 \\
    d\nu_2 &= \alpha_2 (\nu_2 - V_2) dt + \xi_2 \sqrt{\nu_2} dz_4
\end{align*}

(4.41)–(4.44)

where \nu_i = \sigma_i^2, \alpha_i is the rate of mean reversion on the variance, \xi_i is the volatility of the variance and the Wiener processes have the following correlation matrix:

\[
    \rho_2 = \begin{pmatrix}
        1 & \rho_{12} & \rho_{13} & \rho_{14} \\
        \rho_{12} & 1 & \rho_{23} & \rho_{24} \\
        \rho_{13} & \rho_{23} & 1 & \rho_{34} \\
        \rho_{14} & \rho_{24} & \rho_{34} & 1
    \end{pmatrix}
\]

In this case we need to generate four correlated normal variates in order to simulate the four processes (4.41)–(4.44). Then, we simply add the simulation of the variances into the pseudo-code of Figure 4.17. The resulting pseudo-code is shown in Figure 4.19. Figure 4.20 gives a numerical example, the calculations are similar to those for the previous example.
\( \times (-1.3859) = -1.6766 \)

\( 378 - 1 = 17.604 \)

\[ \text{es of CT are accumulated in CT2 = 21844.2. The estimate} \]

\[ \exp(-0.06 \times 1) = 6.9970 \]

\[ \text{general stochastic processes simply simulate the required options of European spread option, but where the variance of square root processes (see equation and variances are given by} \]

\[ \begin{align*}
(4.41) \\
(4.42) \\
(4.43) \\
(4.44)
\end{align*} \]

\[ \text{e, } \xi_i \text{ is the volatility of the option matrix:} \]

\[ \text{es in order to simulate the option of the variances into is shown in Figure 4.19.} \]

\[ \text{similar to those for the} \]

\[ \text{FIGURE 4.19 Pseudo-code for Monte Carlo Valuation of a European Spread Option with Stochastic Volatilities} \]

\[ \begin{align*}
\text{initialize parameters} & \\
\{ \text{K, T, S1, S2, sig1, sig2, div1, div2, alpha1, alpha2, Vbar1, Vbar2, xi1, xi2, rho, r, N, M} \} & \\
\text{precompute constants} & \\
\text{M = 1} \{ \text{no path dependency} \} & \\
\delta t = T/N & \\
\alpha_{1\delta t} = \alpha_{1 \times \delta t} & \\
\alpha_{2\delta t} = \alpha_{2 \times \delta t} & \\
\xi_{1\delta t} = \xi_{1 \times \sqrt{\delta t}} & \\
\xi_{2\delta t} = \xi_{2 \times \sqrt{\delta t}} & \\
\text{lnS1 = ln}(S1) & \\
\text{lnS2 = ln}(S2) & \\
\text{sum CT} = 0 & \\
\text{sum CT2} = 0 & \\
\text{for } j = 1 \text{ to } M \text{ do} \{ \text{for each simulation} \} & \\
\text{lnSt1 = lnS1} & \\
\text{lnSt2 = lnS2} & \\
\left( \begin{array}{c}
\sqrt{\tau_1} = \xi_{1 \times \delta t} \\
\sqrt{\tau_2} = \xi_{2 \times \delta t}
\end{array} \right) & \\
\text{for } i = 1 \text{ to } N \text{ do} \{ \text{for each time step} \} & \\
\text{generate correlated normals( rhoz, z[] } & \\
\text{simulate variances first} & \\
Vt1 = Vt1 + \alpha_{1\delta t}(Vbar1-Vt1) + \xi_{1\delta t}\sqrt{Vt1} z[3] & \\
Vt2 = Vt2 + \alpha_{2\delta t}(Vbar2-Vt2) + \xi_{2\delta t}\sqrt{Vt2} z[4] & \\
\text{simulate asset prices} & \\
\text{lnSt1 = lnSt1 + (r-div1-0.5^2Vt1)*dt + sqrt(Vt1)*stdz[1]} & \\
\text{lnSt2 = lnSt2 + (r-div2-0.5^2Vt2)*dt + sqrt(Vt2)*stdz[2]} & \\
\text{next i} & \\
\text{St1 = exp(lnSt1)} & \\
\text{St2 = exp(lnSt2)} & \\
\text{CT = max( 0, St1 - St2 - K )} & \\
\text{sum CT} = \text{sum CT} + CT & \\
\text{sum CT2} = \text{sum CT2} + CT \times CT & \\
\text{next j} & \\
\text{call value = sum CT/M*exp(-r*T)} & \\
\text{SD = sqrt( (sum CT2 - sum CT*sum CT/M)*exp(-2*r*T)/(M-1) )} & \\
\text{SE = SD/sqrt(M)} & \\
\end{align*} \]
FIGURE 4.20  Numerical Example for Monte Carlo Valuation of a European Spread Option with Stochastic Volatilities

<table>
<thead>
<tr>
<th>K</th>
<th>T</th>
<th>S1</th>
<th>S2</th>
<th>sig1</th>
<th>sig2</th>
<th>div1</th>
<th>div2</th>
<th>alpha1</th>
<th>alpha2</th>
<th>Vbar1</th>
<th>Vbar2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>110</td>
<td>110</td>
<td>0.2</td>
<td>0.3</td>
<td>0.03</td>
<td>0.04</td>
<td>1.0</td>
<td>2.0</td>
<td>0.04</td>
<td>0.09</td>
</tr>
</tbody>
</table>

\[ \begin{align*}
\text{xi1} & \quad \text{xi2} \quad \text{rho12} \quad \text{rho13} \quad \text{rho14} \quad \text{rho23} \quad \text{rho24} \quad \text{rho34} \quad \text{r} \quad \text{N} \quad \text{M} \quad \text{sum}_{-}\text{CT} \quad \text{sum}_{-}\text{CT2} \quad \text{SD} \\
0.05 & \quad 0.06 \quad 0.50 \quad 0.20 \quad 0.01 \quad 0.01 \quad 0.30 \quad 0.30 & \quad 0.06 & \quad 10 & \quad 100 & \quad 747.42 & \quad 22910.0 & \quad 12.4219 \\
\end{align*} \]

\[ \begin{align*}
\text{dt} & \quad \text{sd} \quad \text{alpha1} \quad \text{alpha2} \quad \text{xi1sd} \quad \text{xi2sd} \quad \text{lnS1} \quad \text{lnS2} \quad \text{call\_value} \quad \text{SE} \\
0.1000 & \quad 0.3162 \quad 0.1000 \quad 0.2000 \quad 0.0190 \quad 0.0190 \quad 4.6052 \quad 4.7005 & \quad 7.0389 & \quad 1.2422 \\
\end{align*} \]

\[ \begin{align*}
J = 100 \\
\text{d1} & \quad \text{d2} \quad \text{t} \quad \text{c1} \quad \text{c2} \quad \text{c3} \quad \text{c4} \quad \text{c5} \quad \text{c6} \quad \text{c7} \quad \text{c8} \quad \text{c9} \quad \text{c10} \\
0 & \quad 0.1834 \quad -1.942 \quad 0.0110 \quad -1.3283 \quad -1.6112 \quad -1.1443 \quad -0.6281 \quad 2.3196 \quad -0.6359 \quad 1.3640 \\
0 & \quad -2.3086 \quad 0.8641 \quad 0.3365 \quad -0.6725 \quad -1.1617 \quad 0.5131 \quad 2.1820 \quad 0.8927 \quad 1.3425 \quad 0.7956 \\
0 & \quad 0.8857 \quad -1.4512 \quad 0.5038 \quad -0.3170 \quad 0.7995 \quad -0.7667 \quad 0.3317 \quad -0.6933 \quad 1.4908 \quad 1.3724 \\
0 & \quad -0.3571 \quad 2.8226 \quad 1.3109 \quad 0.5364 \quad -0.3544 \quad 0.4675 \quad -1.4347 \quad 0.2268 \quad 1.1938 \quad 0.3549 \quad -0.9667 \\
0 & \quad 0.8913 \quad -1.2452 \quad 0.9834 \quad -0.6960 \quad -1.1123 \quad 0.2946 \quad -1.2538 \quad 1.2354 \quad -1.7865 \quad 0.3465 \\
0 & \quad 1.5421 \quad -1.4429 \quad -0.4211 \quad -0.7995 \quad -0.3732 \quad 1.1832 \quad -1.5528 \quad 1.6031 \quad -0.4005 \quad 0.8264 \\
0 & \quad 1.1217 \quad 1.2013 \quad 0.3317 \quad 0.7760 \quad 0.8586 \quad -0.0723 \quad 0.7307 \quad 2.1469 \quad -0.2034 \quad 0.1552 \\
0 & \quad 1.1517 \quad -0.477 \quad 0.7936 \quad 1.1499 \quad 1.3035 \quad 0.3684 \quad 1.3327 \quad 1.0725 \quad 1.1206 \quad 2.6165 \\
0.04 & \quad 0.9112 \quad 0.9499 \quad 0.0032 \quad 0.0204 \quad 0.0204 \quad 0.0213 \quad 0.0347 \quad 0.0739 \quad 0.0869 \quad 0.0867 \\
0.09 & \quad 0.9717 \quad 0.0553 \quad 0.1093 \quad 0.0762 \quad 0.0568 \quad 0.0543 \quad 0.0876 \quad 0.1085 \quad 0.1261 \quad 0.1559 \\
\text{lnSt1} & \quad 4.6062 \quad 4.6307 \quad 4.6059 \quad 4.6102 \quad 4.5869 \quad 4.5485 \quad 4.5372 \quad 4.4622 \quad 4.5765 \quad 4.4210 \quad 4.4489 \\
\text{lnSt2} & \quad 4.7005 \quad 4.8295 \quad 4.6859 \quad 4.6384 \quad 4.5738 \quad 4.5451 \quad 4.4058 \quad 4.2581 \quad 4.4216 \quad 4.3724 \quad 4.4709 \\
\end{align*} \]